Singularities of varying light speed cosmologies

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Abstract. We study the possible singularities of isotropic cosmological models that have a varying speed of light as well as a varying gravitational constant. The field equations typically reduce to two dimensional systems which are then analyzed both by dynamical systems techniques in phase space and by applying the method of asymptotic splittings. In the general case we find initially expanding closed models which recollapse to a future singularity and open universes that are eternally expanding towards the future. The precise nature of the singularities is also discussed.

1. Introduction
Varying speed of light (VSL) cosmologies have recently received considerable attention as alternatives to cosmological inflation to provide a different basis for resolving the problems of the standard model (see for example [1] and refs. therein). Instead of adopting the inflationary idea that the very early universe experienced a period of superluminal expansion, in these universes one assumes that light travelled faster in the early universe. In such cosmological models the known puzzles of the standard early universe cosmology are absent at the cost of breaking the general covariance of the underlined gravity theory. We do not enter here into a discussion on the foundations of VSL theories and the conceptual problems arising from the very meaning of varying the speed of light (see [1] for a discussion).

In this paper we report on some preliminary results of our on-going work [2] on the application of the method of asymptotic splittings introduced in [3] to study the singularities that may arise in VSL cosmologies. In particular, we focus here on the VSL model proposed in [4] and further investigated in [5]. An important characteristic of this model is that one assumes minimal coupling at the level of the Einstein equations so that a time-variable $c$ should not introduce changes in the curvature terms in Einstein’s equations in the cosmological frame, Einstein’s equations must still hold. As Barrow points out [4], $c$ changes in the local Lorentzian frames associated with the cosmological expansion, a special-relativistic effect, so that the resulting theory is not covariant and one has to make a specific choice of time coordinate. Choosing that specific time to be comoving proper time the Friedman equations retain their form with $c(t)$ and $G(t)$ varying.
We assume an equation of state of the form \( p = (\gamma - 1)\rho c^2 \), with \( 0 < \gamma \leq 2 \) and we write the Friedman equations with varying \( c(t) \) and \( G(t) \) as follows:

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3} \rho, \tag{1}
\]

\[
\frac{\ddot{a}}{a} = -\frac{8\pi G}{6} (3\gamma - 2) \rho. \tag{2}
\]

Here \( a \) is the scale factor, \( k = 0, +1, \) or \(-1\). Differentiating the first and using the second equation above we obtain the conservation equation

\[
\dot{\rho} + 3\gamma \rho \frac{\dot{a}}{a} = -\frac{\dot{G}}{G} \rho + 3 \frac{k}{2} \frac{c \dot{c}}{4\pi G}, \tag{3}
\]

and setting \( a = x, \dot{a} = y \) we obtain the non-autonomous system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\frac{8\pi G(t)}{6} (3\gamma - 2) x \rho, \\
\dot{\rho} &= -3\gamma \frac{y}{x} \rho - \frac{\dot{G}}{G(t)} \rho + 3 \frac{k}{2} \frac{c(t) \dot{c}(t)}{4\pi G(t)},
\end{align*}
\]

subject to the constraint

\[
\frac{y^2}{x^2} + \frac{kc^2(t)}{x^2} = \frac{8\pi G(t)}{3} \rho.
\]

2. The flat case

For flat \((k = 0)\) models we set \( z = G\rho \) and the system \((4)\) becomes

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\frac{8\pi}{6} (3\gamma - 2) x z, \\
\dot{z} + 3\gamma \frac{y}{x} z &= 0,
\end{align*}
\]

subject to the constraint

\[
\frac{y^2}{x^2} = \frac{8\pi}{3} z.
\]

This system is exactly that which was analyzed using the method of asymptotic splittings in \([3]\), and we rewrite their result. Putting

\[
x = \alpha \tau^p, \quad y = \beta \tau^q, \quad z = \delta \tau^r,
\]

we find the balance

\[
p = \frac{2}{3\gamma}, \quad q = p - 1, \quad r = -2, \quad \beta = \frac{2}{3\gamma} \alpha, \quad \delta = \frac{1}{6\pi\gamma^2}, \quad \alpha = \text{arbitrary}.
\]

Computation of the eigenvalues of the Kovalevskaya matrix \( K = Df(\alpha) - \text{diag}(p) \) yields the values \(-1, 0, \frac{2(3\gamma - 2)}{3\gamma}\). Applying the constraint to the second equation we obtain an integrable system with general solution

\[
\begin{align*}
x &= (At + C)^{2/3\gamma}, \quad y = \frac{2A}{3\gamma} (At + C)^{2/3\gamma - 1}, \\
z &= \frac{A^2}{6\pi\gamma^2} (At + C)^{-2},
\end{align*}
\]
where $A$ and $C$ are constants of integration. We see that the density function always has a finite time singularity. On the other hand, the Hubble parameter blows up in finite time only for $\gamma < 2/3$. It is interesting that the leading-order terms (6) obtained from the calculation of the dominant balance already contain all the information provided by the exact solution (7), an effect due to the fact that the system is weight-homogeneous. We will report on the details of the full analysis of this system elsewhere.

3. Reduction to two dimensions
We assume a power-law dependence of the function $c$, $c = c_0 a^n$, $n \in \mathbb{R}$, $G = \text{constant}$, and use a system of units with $8\pi G = 1 = c_0^2$. We can avoid having to deal with denominators if we set $x = 1/a$, $H = \dot{a}/a$. Then the system (1), (2), (3) becomes

\[
\dot{x} = -xH, \\
\dot{H} = -H^2 - \frac{(3\gamma - 2)}{6} \rho, \\
\dot{\rho} = -3\gamma \rho H + 6knu^{2-2n}H,
\]

and the constraint reads

\[
H^2 + ku^{2-2n} = \frac{1}{3} \rho. \tag{9}
\]

We use the constraint to eliminate $x$, and write the system in the form

\[
\dot{H} = -H^2 - \frac{(3\gamma - 2)}{6} \rho, \\
\dot{\rho} = -(3\gamma - 2n) \rho H - 6nH^3. \tag{10}
\]

From Eq. (10) it follows that the phase space of (10) is the set $\{(H, \rho) \in \mathbb{R}^2 : \rho \geq 0, \rho - 3H^2 > 0\}$ for closed models, and $\{(H, \rho) \in \mathbb{R}^2 : \rho \geq 0, \rho - 3H^2 < 0\}$ for the open ones. Note that the corresponding equations in general relativity have the feature that the conservation equation corresponding to the second equation in (10) is just $\dot{\rho} = -3\gamma \rho H$, which implies that the line $\rho = 0$ is invariant, i.e., the trajectories of the system cannot cross the line $\rho = 0$. On the other hand, here equations (10) without further assumptions do not guarantee that a solution curve starting at a point with $\rho > 0$ will not eventually enter the region with $\rho < 0$, which of course is unphysical.

A first task of the method of asymptotic splittings is to find all possible asymptotic forms $H = \alpha \tau^p$, $\rho = \beta \tau^q$ admitted by the system (10). One balance gives $p = 1$, $q = -(3\gamma - 2n)$ with $\alpha = 1$, $\beta = \text{arbitrary}$. This has K-exponents $(-1, 0)$. A second interesting balance is for $p = -1$, $q = -2$ with coefficients given by

\[
\left(\alpha = \frac{2}{3\gamma}, \beta = \frac{4}{3\gamma^2}\right), \left(\alpha = \frac{1}{1-n}, \beta = -\frac{6n}{(3\gamma - 2)(1-n)^2}\right).
\]

The K-exponents are in this case given by the forms

\[
\left(-1, \frac{2}{3\gamma (3\gamma + 2n - 2)}\right), \left(-1, \frac{3\gamma + 2n - 2}{n - 1}\right).
\]

These results lead to formal series expansions of the solutions in a suitable neighborhood of the finite time singularity, they will be presented elsewhere, [2]. It is interesting that a phase
space analysis shows that only closed models run into a finite time future singularity. In fact the phase portrait of \(10\) with \(n = -1/2, \gamma = 1\) is shown in Figure 1. The dashed line in this Figure separates the closed from the open models. We see that every initially expanding universe starting below this line eventually approaches the origin which corresponds to zero density and expansion rate. On the other hand, initially expanding universes starting above the dashed line eventually contract and \(H \rightarrow -\infty, \rho \rightarrow \infty\) in a finite time. Numerical experiments confirm the above analysis.

4. The general case
The reduction of the three dimensional dynamical system \(8\) to the two-dimensional system \(10\) is not unique. We can use the constraint \(9\) to eliminate \(\rho\) instead of \(u\). However, we shall see that assuming a power-law dependence for \(G\) as well, we can handle both cases at once. In this Section we assume that

\[ c = c_0 a^n, \quad G = G_0 a^m, \quad n, m \in \mathbb{R}, \]

and use a system of units with \(8\pi G_0 = 1 = c_0^2\). Setting again \(x = 1/a, H = \dot{a}/a\), the system \(1\), \(2\), \(3\) can be written as

\[ \dot{x} = -xH, \]
\[ \dot{H} = -H^2 - \frac{1}{6} (3\gamma - 2) \rho x^{-m}, \]
\[ \dot{\rho} = -3\gamma \rho H - m \rho H + 6k \pi x^{2+m-2n} H, \]
subject to the constraint

\[ H^2 + kx^{2-2n} = \frac{1}{3}\rho x^{-m}. \]  \hspace{1cm} (12)

We use the constraint to eliminate \( \rho \), and arrive at the two-dimensional system

\[
\begin{align*}
\dot{x} &= -xH, \\
\dot{H} &= -\frac{3\gamma}{2}H^2 - \frac{(3\gamma - 2)}{2}kx^{2-2n}.
\end{align*}
\]  \hspace{1cm} (13)

Note that from (12) the phase space of (13) is the set

\[
\{ (x, H) \in \mathbb{R}^2 : x \geq 0, H^2 + kx^{2-2n} > 0 \}. \]  \hspace{1cm} (14)

This system is now in a form suitable for both a dynamical systems analysis and an application of the method of asymptotic splittings. Full details will be given elsewhere but we report here on some partial results to give a flavor of the analysis. Putting \( x = \alpha \tau^p \), \( H = \beta \tau^q \) we find a possible balance with \( q = -1, p = -2/3\gamma \) and \( \beta = 2/3\gamma = 1, \alpha = \) arbitrary, acceptable for \( n < 0 \).

The Kovalevskaya eigenvalues for this balance are at \(-1, 0\), compatible with the arbitrariness of \( \alpha \).

In the case where all terms are dominant, the vector field is

\[
\mathbf{f}^{(0)} = \begin{bmatrix} -xH \\ -\frac{3\gamma}{2}H^2 - \left(\frac{3\gamma}{2} - 1\right)kx^{2-2n} \end{bmatrix}
\]

and we find the balance (disregarding the case \( \alpha = 0, \beta = 0 \))

\[
\left(q = -1, \ p = \frac{1}{n - 1}\right), \quad \left(\beta = \frac{1}{1 - n}, \ \alpha^{2-2n} = \frac{2 - 2n - 3\gamma}{(3\gamma - 2)k(1 - n)^2}\right),
\]

with K-exponents given by

\[-1, \ \frac{2 - 2n - 3\gamma}{1 - n}\].

Note that if \( n = -1/2 \) and \( \gamma = 1 \) (dust), or if \( n = -1 \) and \( \gamma = 4/3 \) (radiation) the second eigenvalue is equal to 0.

We may proceed to analyze the system in phase space. First, we write the system it as a single differential equation,

\[
\frac{dH}{dx} = \frac{3\gamma H^2 + (3\gamma - 2)kx^{2-2n}}{2xH}, \quad (15)
\]

and we set \( H^2 = z \) to obtain a linear differential equation for \( z \) which is easily integrable. We find that (15) has the general solution

\[
\begin{align*}
H^2 &= \frac{(3\gamma - 2)k}{2 - 3\gamma - 2n}x^{2-2n} + Cx^{3\gamma} \quad \text{if} \ 3\gamma + 2n \neq 2, \\
H^2 &= \frac{(3\gamma - 2)k}{3\gamma - 2n}x^{3\gamma} \ln x + Cx^{3\gamma} \quad \text{if} \ 3\gamma + 2n = 2,
\end{align*}
\]  \hspace{1cm} (16)

where \( C \) is a constant of integration. Suppose that \( 3\gamma + 2n \neq 2 \). We observe that the leading-order terms of the first decomposition, namely

\[
x = \alpha \tau^{-2/3\gamma}, \quad H^2 = \frac{4}{9\gamma^2} \tau^{-2},
\]
Figure 2. The phase portrait of (13)

reproduce the $C x^{3\gamma}$ part of the solution (16). The leading-order behavior of the last decomposition,

$$x \sim \tau^{1/(n-1)}, \quad H^2 \sim \tau^{-2},$$

recovers the $\frac{(3\gamma - 2)k}{2-3\gamma - 2n} x^{2-2n}$ part of the solution.

The integral curves of (16) allow us to sketch the phase portrait of (13). Let us assume for concreteness that $n = -1, \gamma = 4/3$ (radiation). There are two cases to consider:

- Closed models. The phase space (see (14)) is the half-plane $x \geq 0$. Equation (16) yields $H^2 = -2x^3 + C x^4$, which implies that $C > 0$. Any orbit starting in the first quadrant satisfies $x \geq 2/C$, i.e., there are no solutions approaching the origin. For $C > 0$, any orbit of (13) starting in the first quadrant crosses the $x$-axis at $2/C$ and remains in the fourth quadrant. We conclude that any initially expanding closed universe reaches maximum expansion at some finite time and then recollapse begins, i.e. $H$ approaches $-\infty$ in a finite time. The phase portrait is shown in Figure 2.

- Open models. From (14) we see that the phase space is the set $x \geq 0, \quad H^2 > x^3$. In the first quadrant this set is represented as the area above the dashed line in Figure 2 B. Eq. (16) yields $H^2 = +2x^3 + C x^4$, which implies that for $C > 0$ any orbit starting in the first
quadrant asymptotically approaches the origin. For \( C < 0 \), there are homoclinic curves connecting the origin with itself. However, we must remember that the allowed initial conditions for expanding universes lie above the dashed line in Figure 2. It can be shown that these trajectories also asymptotically approach \((0,0)\). We conclude that any initially expanding open universe remains ever-expanding.

In all, we find that VSL cosmological models may share many interesting dynamical characteristics not fully present in the more conventional models so that a more detailed study of the singularities in these universes is worthwhile as a future challenge.

References