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Generalized cosmic no-hair theorems

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We show that the quasi-exponential solution $\exp(Bt-At^2)$ of $R+\alpha R^2$ gravity is an attractor for all homogeneous and isotropic solutions of higher order gravity theories derived from a lagrangian that is an arbitrary analytic function of the scalar curvature R. This indicates that a generalized form of the cosmic no-hair conjecture is valid in the framework of higher order gravity theories.

1. Recently much theoretical interest has been focused on the question of the naturalness of the inflationary scenario. One can address this question in many different ways, but the essence of it can be stated as follows: Does the inflationary phase in the evolution of the universe proceed from very general initial conditions? In other words is the inflationary solution an "attractor" to all possible pre-inflationary states of the universe?

A closely related issue is what is known as the cosmic no-hair conjecture [1]: All expanding-universe models with positive cosmological constant asymptotically approach the de Sitter solution. Let us point out from the beginning that (1) a precise version of this conjecture is very difficult to formulate, mainly because of the vagueness associated with the terms "asymptotic approach" and "expanding universe"; (2) there is no general proof (or disproof) of this conjecture; and (3) some counter-examples exist of the form "initially expanding-universe models recollapse to a singularity" without ever becoming de Sitter type universes (see ref. [2]).

However, Wald [3] has proven the following simplified version of the cosmic no-hair conjecture: All Bianchi-type (homogeneous but anisotropic) models, except Bianchi IX, with a positive cosmological constant asymptotically approach the de Sitter solution. Note that for the validity of Wald's result (and in fact for the validity of all cosmic no-hair conjectures) the stress tensor of any matter fields is constrained to satisfy the dominant and strong energy conditions [4]. Further, some indication that the cosmic no-hair conjecture may be true in some inhomogeneous models has been given in ref. [5]. In fact Starobinski in ref. [6] has shown that inhomogeneous spacetimes that do inflate are of the most general type.

One may use the conformal equivalence theorem (cf. ref. [7]) freely to prove similar results in the context of higher-order gravity theories derived from a lagrangian of the general form

$$L_{\mathrm{HOG}}=f(R)\,, \qquad (1)$$

where $f(R)$ is an arbitrary analytic function of the scalar curvature R, which presumably is important during the pre-inflationary phase. However, the present incomplete status of the proof of the cosmic no-hair theorem in the setting of general relativistic cosmology does not allow one to conclude unambiguously that a finite period of inflation fits naturally into the more general cosmological framework we are considering.

On the other hand it is a very important result [6,8,9] that $f(R)$ theories without a cosmological constant still admit a near-de Sitter solution corresponding to a quasi-exponential expansion of the universe. This solution, in the case of the quadratic lagrangian $f(R)=R+\alpha R^2$, has the form

$$a_0(t)=\exp(Bt-At^2) \qquad (2)$$

with A, B constants and $A=1/72\alpha<0$.

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2. In this paper we present an indirect proof of a generalized form of the cosmic no-hair theorem valid in the framework of higher order gravity theories derived from the lagrangian (1). The conjecture is as follows:

Conjecture 1. All solutions of the higher order gravity theory derived from the lagrangian (1) with (a) a metric which can be written in a synchronous form and (b) a stress–energy tensor which satisfies the strong and dominant energy conditions asymptotically approach the quasi-de Sitter solution (2).

In the following we assume the validity of (a) and (b) in the above conjecture. This is a reasonable assumption to make as can be easily checked by transforming the lagrangian (1) into the equivalent system of Einstein gravity plus a scalar field matter source with a particular self-interaction potential [7]. Then one easily sees that the stress–energy tensor of that scalar field satisfies the assumption (b) appearing in our conjecture.

Our proof is based on a perturbation analysis of the solutions (2). This method is similar in spirit to the analysis presented by Barrow and Ottewill [9] in the sense that they considered the stability of de Sitter solutions to higher order gravity of the exponential form $a_0 = \exp(H_0 t)$ and were able to show that these solutions are stable in higher order gravity provided certain conditions hold. For example, in quadratic lagrangian theories of the form $R + \alpha R^2$ these solutions are stable if $\alpha < 0$ and unstable if $\alpha > 0$.

Since (2) are not solutions of general relativity, a stability analysis of (2) in $f(R)$ theories is essentially equivalent to examining the validity of the cosmic no-hair conjecture for solutions of the form

$$a(t) = \exp(Bt - At^2)[1 + \varepsilon(t)]$$
$$|\varepsilon(t)| \ll 1 . \tag{3}$$

If these solutions turn out to have a stable regime in higher order gravity, then in that regime a_0 from (2) will be an attractor for all solutions of the form (3) and the cosmic no-hair theorem will follow immediately.

3. We now derive the general perturbation equation for the quasi-de Sitter solution (2) in the framework of $f(R)$ theory. To simplify the calculation we set $\nu = B - At$. Then proceeding similarly as in refs. [9,10] and after some manipulation we find that the perturbation $\varepsilon(t)$ for every $t > 0$ satisfies the equation [10]

$$f''_0 \dddot{\varepsilon} + \left[3 f''_0 \nu + A\left(48 f'''_0 \nu + \frac{2 f''_0}{\nu} \right) \right] \ddot{\varepsilon}$$
$$- \tfrac{1}{3} [f'_0 + 12\nu^2 f''_0 - 3A(192\nu^2 f'''_0 - 8 f''_0)] \dot{\varepsilon}$$
$$= 0 , \tag{4}$$

where $f_0 = f(R_0)$ etc. (cf. refs. [9,10] for this notation). In the case where $A = 0$ this equation reduces to the perturbation equation that describes the evolution of perturbations to the de Sitter space of general relativity [9]. By solving eq. (4) we can decide whether or not the quasi-de Sitter space given above is a stable solution of the $f(R)$ theory.

Setting $\dot{\varepsilon} = \phi$, dividing both sides by f''_0 and putting $\lambda_1^2 = f'_0 / 3 f''_0$ and $\lambda_2^2 = f'''_0 / 3 f''_0$ eq. (4) becomes

$$\ddot{\phi} + 3\left((B - 2At)(1 + 48A\lambda_2^2) + \frac{2A}{3(B - 2At)} \right) \dot{\phi}$$
$$- [\lambda_1^2 + 8A + 4(B - 2At)^2 (1 - 144A\lambda_2^2)] \phi$$
$$= 0 . \tag{5}$$

Now with $B - 2At = \tau$, $\phi(t) = u(\tau)$ and $' \equiv \mathrm{d}/\mathrm{d}\tau$, eq. (5) becomes

$$\tau u'' + \left(\frac{-3\alpha_1}{2A} \tau^2 - 1 \right) u' + \left(\frac{-k}{4A^2} \tau^3 - \frac{\gamma}{4A^2} \tau \right) u$$
$$= 0 , \tag{6}$$

where $\alpha_1 = 1 + 48A\lambda_2^2$, $\gamma = \lambda_1^2 + 8A$ and $k = 4(1 - 144A\lambda_2^2)$. Under the transformation

$$u(\tau) = \tau \exp\left(\frac{3\alpha_1}{8A} \tau^2 \right) w(\tau) \tag{7}$$

eq. (6) becomes

$$\tau^2 w''(\tau) + \tau w'(\tau)$$
$$+ \left[\tau^4 \left(- \frac{k}{4A^2} - \frac{9\alpha_1^2}{16A^2} \right) - \frac{\gamma}{4A^2} \tau^2 - 1 \right] w(\tau)$$
$$= 0 . \tag{8}$$

At this point some remarks are in order. There is no known solution to eq. (8) [11]. Nevertheless, we shall show that there is no loss of generality in the overall result if we disregard either the τ^2 or the τ^4 terms in eq. (8). We consider these two cases separately.

(i) The removal of the τ^4 term in (8) requires

$$\frac{k}{4A^2}+\frac{9\alpha_1^2}{16A^2}=0\,, \tag{9}$$

and in this case eq. (8) becomes a Lommel equation [12]. Note in passing that condition (9) means that λ_2^2 and α are restricted so that

$$\frac{\lambda_2^2}{\alpha}=\frac{5}{2}\,. \tag{10}$$

Condition (10) in the case of the gravitational lagrangian (1) means that the coefficients α_i, $i=1,\ldots,n$ of the powers of R in the Taylor expansion around R_0 are constrained to the hypersurfaces (10) on the n-dimensional space spanned by the α_i.

Thus solving (8) and taking into account (10) we find

$$\begin{aligned}\varepsilon(t)&=\frac{c_1}{2A}\int_0^\tau x\exp\left(\frac{x^2}{A}\right)J_1\left(\frac{x}{|2A|}\sqrt{-\lambda_1^2-8A}\right)\mathrm{d}x\\&+\frac{c_2}{2A}\int_0^\tau x\exp\left(\frac{x^2}{A}\right)N_1\left(\frac{x}{|2A|}\sqrt{-\lambda_1^2-8A}\right)\mathrm{d}x\\&+c_3\,,\\ \tau&=B-2At\,,\end{aligned} \tag{11}$$

where J_1 and N_1 are the Bessel functions of the first and second kind respectively. The asymptotic behaviour of the perturbation $\varepsilon(t)$ as $t\to\infty$ $(\tau\to\infty)$ depends on the sign of $-\lambda_1^2-8A$ in (11). We quote the results:

(I) $-\lambda_1^2-8A>0$. For $t\to\infty$ $(\tau\to\infty)$ and due to the obvious continuity of the integrals with respect to τ and the asymptotic behaviour of $J_1(z)$, $N_1(z)$ [#1] we observe that the $\lim_{t\to\infty}\varepsilon(t)$ exists since $A<0$. We note that this is an important conclusion as it is in accordance with the basic physical assumption which must be imposed on A (see eq. (2) and following). Therefore, with $\tau=B-2At$ we find from (11), by using standard formulae from ref. [12],

$$\begin{aligned}\lim_{t\to\infty}\varepsilon(t)&=\frac{c_1\sqrt{\pi A(\lambda_1^2+8A)}}{32A}\,{}_1F_1\left(\tfrac{3}{2},2,\frac{\lambda_1^2+8A}{-16A}\right)\\&+(-A)^{1/2}c_2\exp\left(\frac{-(\lambda_1^2+8A)}{32A}\right)\left(\sqrt{-\lambda_1^2-8A}\right)^{-1}\\&\times W_{1/2,1/2}\left(\frac{\lambda_1^2+8A}{16A}\right)\\&+c_3\,,\end{aligned} \tag{12}$$

where ${}_1F_1(\;)$ and $W_{1/2,1/2}(\;)$ are the confluent hypergeometric and Whittaker functions respectively.

We conclude that $|\varepsilon(t)|$ is bounded for all t and thus the constants c_1, c_2 and c_3 in (12) can be chosen such that $\lim_{t\to\infty}\varepsilon(t)=0$. Alternatively, for sufficiently large t in eq. (11), the perturbation $\varepsilon(t)$ can be made arbitrarily small by choosing c_1, c_2 and c_3 appropriately.

(II) $-\lambda_1^2-8A<0$. Here J_1, N_1 in (11) are replaced by the modified Bessel functions I_1, K_1 respectively. The integrals in (11) converge for $\tau\to\infty$ since $A<0$ and

$$I_1(z)\sim\frac{\exp(z)}{\sqrt{z}}\,,\quad K_1(z)\sim\frac{\exp(-z)}{\sqrt{z}}\,,\qquad z\to\infty \tag{13}$$

and so in this case we have again that $|\varepsilon(t)|$ is bounded for all t and also that $\lim_{t\to\infty}\varepsilon(t)=0$, provided c_1, c_2 and c_3 are chosen appropriately.

(III) $\lambda_1^2+8A=0$. Similarly we find

$$\begin{aligned}\varepsilon(t)&=\frac{c_1}{2A}\int_0^\tau x\exp\left(\frac{x^2}{A}\right)\exp(x)\,\mathrm{d}x\\&+\frac{c_2}{2A}\int_0^\tau x\exp\left(\frac{x^2}{A}\right)\exp(-x)\,\mathrm{d}x\\&+c_3\,,\\ \tau&=B-2At\,.\end{aligned} \tag{14}$$

Obviously the integrals in (14) converge since $A<0$. The result is again that $\lim_{t\to\infty}\varepsilon(t)=0$ can be ensured for appropriate c_1, c_2 and c_3.

(ii) Removing the τ^2 term and following the same analysis as in (i) above yields $\lim_{t\to\infty}\varepsilon(t)=0$, $|\varepsilon(t)|$

[#1] As $z\to\infty$ one has $J_1(z)\sim z^{-1/2}\cos(z-3\pi/4)$, $N_1(z)\sim z^{-1/2}\times\sin(z-3\pi/4)$ (cf. ref. [12]).

being bounded for all t since $A<0$. In this case too the coefficients α_i in the gravitational lagrangian are constrained to an $(n-1)$-hypersurface.

Before we proceed two important points must be stressed:

(1) Keeping just one of the τ^4 or τ^2 term in (8) does not affect the overall result and simply constraints the coefficients α_i to move on a hypersurface.

(2) Moreover, the basic condition $A<0$ appearing in eq. (2), which is the only constraint needed to ensure that $|\varepsilon(t)|$ is bounded for all t, persists irrespective of whether we remove the τ^4 or the τ^2 term in (8). Consequently, one clearly observes that, as in fact we have claimed, indeed no loss of generality has incurred through the analysis in (i) and (ii).

4. As a consistency check consider, for example, the cubic lagrangian theory

$$f(R)=R+\alpha R^2+\beta R^3 . \tag{15}$$

This choice leads to some interesting cosmological properties relating to the self-regenerating mechanism in stochastic inflation [13]. From (10) and since $\alpha<0$ ($A=1/72\alpha$) we find ($R_0=12B^2$)

$$\lambda_2^2=\frac{6\beta}{6\alpha+18\beta R_0}=\frac{\beta}{\alpha+36\beta B^2}<0 , \tag{16}$$

and conditions (10), (16) require

$$\beta=-\frac{5\alpha^2}{2(1-90\alpha B^2)} , \quad \alpha<0 . \tag{17}$$

Thus our conjecture is justified in the case of the lagrangian (15) if $\alpha<0$ and the constraint (17) holds between α and β.

5. The above analysis shows that the generalized form of the cosmic no-hair conjecture put forward in this paper may be valid in the framework of higher order gravity theories which are derived from the gravitational lagrangian (1). This means that all solutions of higher order gravity theories of the form (3) are attracted by the solution (2) and, as time passes, the solutions (3) eventually settle down to a quasi-stationary state described by (2). This result belongs to the "world" of higher order gravity alone and does *not* rely on knowledge of the cosmic no-hair theorem in general relativity. Since inflationary regimes in higher order gravity theories do not seem to depend on phase transitions [8], the theorem proved here implies that the response of the quasi-de Sitter phase (2) to small gravitational perturbations of homogeneous and isotropic type is good and gravitational distortions typically do not affect its stability if the coefficient of the R^2 term in the gravitational lagrangian comes in with a negative sign. Recall that $\alpha<0$ is also needed for the stability of Friedman solutions, the existence of wormhole solutions and the absence of tachyons in higher order gravity theories (cf. refs. [9,10] and references therein). In this case we showed that homogeneous and isotropic gravitational perturbations of the metric tensor tend to zero asymptotically as $t\to\infty$. According to our picture universes that are close to the quasi-de Sitter phase (2) when inflation commences near the Planck time in higher order gravity stay close during the subsequent inflationary phase. It remains to be seen (possibly along the lines of ref. [14]) whether or not anisotropic or inhomogeneous perturbations tend asymptotically to zero as $t\to\infty$. We leave these more general cases to a future publication.

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