CHAOTIC BEHAVIOUR IN HIGHER-ORDER GRAVITY THEORIES

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We show that the chaotic dynamical behaviour displayed by diagonal Bianchi type IX metrics in general relativity does not occur on approach to the singularity in higher-order lagrangian theories of gravity. However, chaotic behaviour does occur in the more general non-diagonal type IX models in these theories. An interpretation of these results in terms of the hamiltonian potential picture of the type IX evolution is given.

There has been extensive study of the conditions under which chaotic behaviour can and must appear on approach to a cosmological singularity in general relativity [1]. The evolution of the Einstein equations for a spatially homogeneous vacuum universe of Bianchi type VIII or IX in (3+1) spacetime dimensions has the property that evolution towards an all-encompassing Weyl scalar curvature singularity occurs via a chaotically unpredictable sequence of oscillations [2]. If the dynamics are discretized by the construction of a Poincaré return mapping then, remarkably, the smooth invariant measure can be found analytically [3]. Various more detailed analyses have been performed to determine the nature of this chaotic behaviour and how its presence is linked to the number of spatial dimensions in the universe [4–6]. If the spacetime metric is diagonal and has the product manifold structure of Kaluza–Klein type, with uncoupled internal and external manifolds, then chaotic behaviour is only possible in vacuum (or perfect fluid, or gauge field filled) universes with three spatial dimensions. If the spatially homogenous manifold is of non-product type and off-diagonal metric components are included then chaos occurs in models whose spatial dimension lies in the inclusive range from three to nine. Chaotic behaviour can disappear in ten or more space dimensions.

On approach to a cosmological singularity we would expect higher-order curvature corrections to those included in the linear Einstein–Hilbert action to play the dominant role in determining the evolution of the spacetime geometry. Moreover, various approaches to the quantization of gravity and the unification of the forces of Nature give rise to such corrections to general relativity [7]. As a result, it is important to determine whether the chaotic behaviour found in general relativity is retained on approach to a singularity (if indeed there still is a singularity), if it disappears, or if a new type of chaos emerges.

In an earlier analysis Barrow and Sirousse-Zia [8] investigated the dynamics of the diagonal Bianchi type IX universe in three space dimensions derived from a scale-invariant quadratic lagrangian, \( L = R^2 \), where \( R \) is the scalar four-curvature. It was found that there is no chaotic behaviour on approach to the singularity. Here, we shall extend this analysis to the more general quadratic lagrangian case with \( L = R + \alpha R^2 \), \( \alpha \) constant, in both diagonal and non-diagonal Bianchi IX universes and show how the results obtained enable one to decide whether or not chaos arises in gravity theories derived from a gravitational lagrangian that is an arbitrary analytic function of \( R \). We shall approach the problem using two different methods: the first, by a direct analysis of the Bianchi type IX field equations in the higher-order gravity theory; the second, by exploiting the conformal equivalence theorem [9] which reduces pure gravity theories derived from an \( f(R) \) lagrangian to general relativity plus a scalar field matter source possessing some self-interaction potential. We shall confine our attention to four-dimensional spacetimes.
The metric of the spatially homogeneous universe of Bianchi type IX has the form \[1,2,\]
\[ds^2 = dt^2 - \gamma_{\alpha\beta}(x)\sigma^\alpha(x)\sigma^\beta(x), \quad 1 \leq \alpha, \beta \leq 3,\]
where \(\sigma^\alpha(x)\) are the SO(3)-invariant differential forms which generate the homogeneous space of Bianchi type IX. They satisfy \(d\sigma^\alpha = \epsilon^{\alpha\beta\gamma}\sigma^\beta \wedge \sigma^\gamma\) where \(\epsilon^{\alpha\beta\gamma}\) is the completely antisymmetric tensor of rank 3. The time dependence of the spacetime geometry (1) is carried by the three-metric tensor \(\gamma_{\alpha\beta}(t)\).

If the gravitational lagrangian, \(L\), is assumed to be an arbitrary analytic function of the scalar curvature, \(R\),
\[L = f(R),\]
then, by variation of the associated action with respect to the four-dimensional metric \(g_{ab}\) we obtain the field equations \((0 \leq a, b \leq 3)\),
\[0 = R_{ab}f' - \frac{1}{2}g_{ab}\nabla_a\nabla_b f' + g_{ab}\Box f',\]
where \(f' = \partial f/\partial R\), \(\nabla_a\) is the covariant-derivative and \(\Box = g_{ab}\nabla^a\nabla^b\).

If we specialize to the quadratic lagrangian \(L_2 = R + \alpha R^2\), \(\alpha\) constant, \((4)\),
then the field equations (3) reduce to
\[2\alpha\nabla_a\nabla_b R - R_{ab}(1 + 2\alpha R) + 2\alpha g_{ab}\Box R + \frac{1}{2}g_{ab}(R + \alpha R^2) = 0.\]
The trace of (5) is
\[6\alpha \Box R = R.\]
We take the metric to have the Bianchi type IX form (1) with diagonal \(\gamma_{\alpha\beta}\) so, introducing three orthogonal scale-factors \(a_\alpha(t)\), we can write
\[\gamma_{\alpha\beta} = \text{diag}[a_1^2(t), a_2^2(t), a_3^2(t)].\]
Defining three expansion rates, \(h_\alpha(t)\), by \(\dot{a}_\alpha = a_\alpha \frac{d}{dt}\)
\[h_\alpha(t) \equiv \frac{\dot{a}_\alpha}{a_\alpha} : h \equiv \sum_{\alpha=1}^{3} h_\alpha,\]
the field equations (5), (6) become the ordinary differential equations
\[2\alpha h R + R_0 0(1 + 2\alpha R) - \frac{1}{2}(R + \alpha R^2) = 0,\]  
\[2\alpha h_1 R = R_1 1(1 + 2R) + \frac{1}{2}R + \frac{1}{4}\alpha R^2 = 0.\]

Here, (10) is the (1-1) component of (5). The (2-2) and (3-3) have forms derivable from (10) by cyclic permutations of (123). The eqs. (9), (10) can be expressed in terms of the scale-factors \(a_\alpha(t)\) by using
\[-R_0 0 = \frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3},\]
\[-R_1 1 = \frac{(\dot{a}_1 a_2 a_3)}{a_1 a_2 a_3} + \frac{1}{2a_1 a_2 a_3^2} [a_1^4 - (a_2^2 - a_3)^2],\]
and where \(R_2, R_3\) are obtained from (12) by cyclic permutations.

Let us examine first the possibility of non-chaotic, monotonic, power-law asymptotes to the system (9)–(12). We look for asymptotes of the form \(a_\alpha(t) = t^p_\alpha\), \((13)\)
as \(t \to 0\). For simplicity we define constants \(q\) and \(r^2\) by
\[
q = \frac{3}{\sum_{\alpha=1}^{3} p_\alpha}, \quad r^2 = \sum_{\alpha=1}^{3} p_\alpha^2.
\]
Using (13)–(15) in (9)–(12) we obtain for the (1–1) equation (and the others by cycling indices), to leading order, as \(t \to 0\),
\[
\alpha t^4 [4p_1(r^2 + q^2 - 2q) - 2p_1(q - 1)(r^2 + q^2 - 2q) + \frac{1}{2}(r^2 + q^2 - 2q)^2]
+ \frac{1}{2} [p_1(q - 1) + \frac{3}{2}(r^2 + q^2 - 2q)] + [***]
= 0,
\]
where the additional higher-order terms \([***]\) are dominated by
\[
[***] = -(q^2 + r^2 - 2q)t^{-2(q - 2p_1 + 1) + \frac{1}{2}t - 4(q - 2p_1) + 2}
+ t^{-2(q - 2p_1)}[4p_1^2 - p_1(q - 1) + \frac{1}{2}(r^2 + q^2 - 2q) + \frac{1}{2}]
+ t^{-2(p_1 + 1)}[4p_1^2 - 2p_1(q - 1) - (r^2 + q^2 - 2q)]
- t^{-2(q - p_1)} + t^{-2(q - p_1 + 1)} + \frac{3}{2}t^{-2(2p_1 - q) + 2}
+ \frac{1}{2}t^{-4(q - 2p_1)} + \frac{1}{2}t^{-2(q - 2p_1 + 1) + \frac{1}{2}t - 2p_1}
- \frac{3}{2}t^{-4p_1 - \frac{1}{2}t - 2(q - 2p_1 + p_1 - 1)},
\]
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where the last term is a sum of terms over \((\alpha, \beta)\)
where \(1 \leq \alpha 
eq \beta \leq 3\) in the last term. These “additional” terms, \(\{***\}\), are those generated by the non-newtonian anisotropic curvature of the type IX geometry. In the general relativistic \((L = R)\) case they are responsible for generating the chaotic behaviour.

If we ignore these terms for a moment and examine the remaining asymptotic terms in (16) they create asymptotic behaviour of the simpler (non-chaotic) Bianchi type I universe on approach to a singularity at \(t=0\). To deduce this type I behaviour we set the leading-order term to zero both in (16) and in the asymptotes derived for the \((2-2)\) and \((3-3)\) equations by cycling. A solution requires

\[
r^2 = q(2-q),
\]

so \(0 < q < 2\). When \(q = r^2 = 1\) we recover the well-known vacuum Kasner asymptote and we see that this is also the required solution in the general relativity case \((\alpha = 0)\) when the additional terms are neglected. In general, (18) requires the pairwise products of the \(p_\alpha\) to satisfy

\[
\sum_{\alpha \neq \beta} p_\alpha p_\beta = q(q-1) > 0,
\]

and hence, unlike in the \(q = r^2 = 1\) case, all the \(p_\alpha\) can be positive. This situation leads to a stable monotonic approach of all three scale-factors to the singularity and no chaotic behaviour. The indices \(p_\alpha\) are permuted ergodically until a configuration with all three \(p_\alpha > 0\) results and this is asymptotically stable as \(t \to 0\).

If we now examine the “additional terms” in (16), which are the contributions from the anisotropic curvature in the type IX geometry, then we can see that non-chaotic behaviour persists. It is clear that the effect of adding the \(\alpha R^2\) term to the gravitational lagrangian is to alter the vacuum Mixmaster behaviour near the singularity to that found in general relativity in the presence of a free scalar field. We recall that the solution for the Bianchi type I metric in the presence of such a source has the form \([10-12]\)

\[
a = t^{p_1}, \quad b = t^{p_2}, \quad c = t^{p_3}, \quad \varphi = \varphi_0 \ln t,
\]

\[
\sum_{\alpha=1}^3 p_\alpha = 1, \quad \sum_{\alpha=1}^3 p_\alpha^2 = 1 - Q^2, \quad Q^2 < \frac{1}{3}.
\]

Hence, the \(p_\alpha\) span the overlapping ranges

\[-\frac{1}{7} \leq p_\alpha \leq \frac{1}{7}, \quad 0 \leq p_2 \leq \frac{1}{7}, \quad \frac{1}{7} \leq p_3 \leq 1,\]

and it is possible for all three \(p_\alpha\) to be positive when \(Q^2 \neq 0\). In this situation chaotic oscillations eventually cease.

Elsewhere we have shown that there is a conformal relationship between vacuum gravity theories derived from a lagrangian of the form \(L = f(R)\) and general relativity with a scalar field–matter source. This scalar field has a potential, \(V(\varphi)\), determined by the form of \(f(R)\). In the case where \(L = f(R)\) the vacuum field eqs. (3) can be suggestively rewritten (Barrow et al. [7,9]),

\[
f'(R_{ab} - \frac{1}{2} R g_{ab}) + \frac{1}{2} g_{ab}(R f' - f) - \nabla_a \nabla_b f' + g_{ab} \square f' = 0.
\]

Under the conformal transformation \(g_{ab} \to \tilde{g}_{ab}\) where \([9]\)

\[
\tilde{g}_{ab} = [f'(R)] g_{ab},
\]

the field equations become

\[
R_{ab} - \frac{1}{2} \tilde{g}_{ab} R = \nabla_a \varphi \nabla_b \varphi - \frac{1}{2} \tilde{g}_{ab} \nabla_c \varphi \nabla^c \varphi
- \frac{1}{2} \tilde{g}_{ab} (f')^{-2} (f'R - f),
\]

where we have introduced the definition

\[
\varphi = \frac{3}{\sqrt{2}} \ln f'(R).
\]

The last term in (25) is the potential of a self-interacting scalar field, \(\varphi\), with potential \([9]\)

\[
V(\varphi) = \frac{1}{2} (f')^{-2} (f'R - f).
\]

If \(f(R)\) is an analytic function of the form

\[
f(R) = R + \sum_{n=2}^k a_n R^n,
\]

then \(V(\varphi)\) has the form

\[
V(\varphi) = A_1 c^{-8\sqrt{3}} + \sum_{n=2}^k A_n c^{-8\sqrt{3}(1 - 8\varphi)^n},
\]

with \(A_n\) constants. In order to show that there is stable, non-chaotic behaviour on approach to the singularity we need to demonstrate that the solution (20)–(22), which holds when \(V(\varphi) = 0\), is stable as \(t \to 0\) when \(\varphi \to -\infty\). We can see that as \(\varphi \to -\infty\)
V(0) \sim e^{-8\phi/\sqrt{3}}.

But since \( \phi = Q \ln t \) with \( Q^2 < 2/3 \) this means that \( V(\phi) \) can diverge no faster than

\[ V(\phi) \sim t^{-8\phi/\sqrt{3}} \sim t^{-16/9} \]

and hence the potential is negligible compared to the kinematic terms \( \frac{1}{2} \dot{\phi}^2 \sim t^{-2} \) as \( t \rightarrow 0 \). Thus, there exists a stable non-chaotic Kasner-like asymptote with all \( p_\alpha \) positive in the class of higher-order lagrangian theories generated by (4) when the metric is diagonal.

It is known that the inclusion of off-diagonal terms in the Bianchi type IX universe can have an important effect upon its chaotic behaviour [10]. Demaret et al. [5] have shown that chaotic behaviour, although absent in diagonal type IX universes with spatial dimension between four and nine, is re-established when off-diagonal metric terms are included. (Chaos is absent in all cases when the spatial dimension exceeds nine.) We find a similar effect of non-diagonality in the higher-order lagrangian theory. In the non-diagonal case the type IX universe is chaotic in higher-order gravity theories.

To understand this result suppose that we reconsider the conformally transformed problem of general relativity with scalar field. As a result of the conformal relationship the conformally-related metric (24) will also contain off-diagonal terms; and on approach to the initial singularity the potential will be negligible with respect to the kinetic terms, as discussed above. Now, following the standard prescription of Kaluza-Klein we can represent the four-dimensional Bianchi type IX model with such a scalar field as a five-dimensional spacetime (see Belinskii and Khalatnikov [10] for a detailed discussion) with the structure

\[
g_{AB} = \begin{pmatrix} g_{ab} & A_b \\ A_a & \varphi \end{pmatrix}, \quad 0 \leq A, B \leq 4, \quad (32)
\]

where \( g_{ab} \) is the four-dimensional metric, \( A_b = g_{b4} \) and the scalar field \( \varphi = g_{44} \). \( A_b \) is the gauge field of the \( U(1) \) symmetry. The vector field \( A_t \) contributes the off-diagonal metric terms. Thus, including the off-diagonal terms in the type IX metric is equivalent to considering a five-dimensional spacetime. The associated action includes the four-dimensional Einstein action, a Maxwell term for the \( A_b \) contribution and the kinetic term for \( \varphi \). We require that \( A_b \) be a function of \( \varphi \) in order that there is mixing between the diagonal and off-diagonal terms. This disrupts the stability of the Kasner asymptote (20)–(22) which was established by the diagonal terms. Now it was first shown by Belinskii and Khalatnikov [10] that the general relativistic type IX universe with scalar and vector field source is chaotic. Hence, we know that the non-diagonal type IX universe is chaotic in the quadratic lagrangian theory (4) and also in those theories with polynomial lagrangians of the form (28).

It is instructive to give a qualitative interpretation of this result in the hamiltonian potential picture of the type IX evolution introduced by Misner [2] and developed further by Ryan [13]. In the diagonal vacuum type IX model the Einstein equations are a hamiltonian problem for the motion of a point inside an essentially equilateral triangular potential whose walls expand at constant velocity and \( t \rightarrow 0 \) as the singularity is approached. The speed of the universe point relative to that of the walls is such that the point will always be able to catch a wall. The infinite of sequence of collisions that the point makes with the walls is chaotically unpredictable. It is ergodic, strong mixing and isomorphic to a Bernoulli shift and a variety of dynamical invariants can be calculated exactly for the discrete dynamics of the associated Poincaré map. When the scalar field is added, it slows the speed of the universe point relative to that of the walls (in effect by the the change in the form of eq. (21) to allow \( Q^2 \) to be non-zero) so that the point can only catch the wall towards which it is moving if it is not moving at too oblique an angle relative to it. However, after a few collisions with the potential walls the universe point inevitably finds itself moving so obliquely that it never again catches a wall. Its asymptotic behaviour is thus that of a model with no potential. That is, the Kasner solution of the form (20)–(22) and there is no chaotic behaviour. The situation changes again with the addition of non-diagonal metric terms. When off-diagonal terms are added to the type IX metric (1) with time-independent forms \( \sigma^\alpha \) we can carry out a linear transformation to a non-canonical (time-dependent) basis

\[
\sigma^\alpha \rightarrow \sigma'^\alpha = O_{\beta}^\alpha(t) \sigma^\beta,
\]

which diagonalizes the metric in this moving frame.
The effect of the non-diagonal terms is now included in a time-dependence carried by the $\sigma^{\alpha \nu}$ and this causes the potential triangle to rotate relative to the universe point. This allows the infinite chaotic sequence of collisions between the universe-point and the walls to be re-established.

In summary: we have established that in quadratic lagrangian theories of gravity, the chaotic dynamical behaviour displayed by the general relativistic cosmological models of Bianchi type IX on approach to the singularity does not occur when the metric is diagonal. After a finite number of oscillations a stable monotonic contraction of all three orthogonal scale-factors occurs. This confirms the earlier analysis of the pure $R^2$ lagrangian case \[8\]. However, chaotic behaviour is present in the more general case when the type IX metric includes non-diagonal terms. These results hold for lagrangian theories of gravity of polynomial form (28) containing a term linear in the four-curvature, and will also apply to metrics of Bianchi type VIII. They have a simple heuristic interpretation in terms of the hamiltonian potential picture of the type IX metric. It is interesting to compare them with those obtained in general relativity in arbitrary spatial dimensions by Demaret et al. \[5\] (see Barrow, Demaret and Henneaux \[6\] for a detailed survey), where the presence of off-diagonal terms in the metric is also responsible for the re-establishment of a stable sequence of chaotic oscillations on approach to the initial singularity.

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References


