VARIATIONAL STRUCTURES AND COSMOLOGICAL DYNAMICS IN HIGHER ORDER GRAVITY

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Στον Νικόλαο Χατζησάββα, φίλο και δάσκαλο.

Abstract

In the first part of this thesis we analyse the variational structure of arbitrary nonlinear Langrangian theories of gravity. After a critique on the traditional Palatini variation we exploit the consequences of adopting the so-called constrained Palatini variation and prove a general result of equivalence of the constrained Palatini variation to the usual Hilbert method for most of the higher order Lagrangians. We apply this theorem to $f(R)$ theories in the extended framework of Weyl geometries and study the conformal structure of the so-formed theories. We prove a non-trivial generalization of the conformal equivalence theorem valid for arbitrary $f(R)$ Lagrangians in Weyl geometry. In particular, previous results valid in the Riemannian framework appear naturally as special cases of this general result in the limit when Weyl geometries tend to Riemannian ones.

In the second part we study in detail the isotropization problem in the context of $f(R)$ theories. In the framework of inflation, we prove the cosmic no-hair conjecture for all orthogonal Bianchi cosmologies with matter in the $R + \beta R^2 + L_{matter}$ theory. The proof is given in the conformal frame with the scalar field, that has the usual self-interacting potential, in the presence of the conformally related matter fields. We show in particular that the Bianchi IX universe asymptotically approaches de Sitter space provided that, initially, the scalar three-curvature does not exceed the value of the potential of the scalar field associated with the conformal transformation. We present a generalization of the Collins-Hawking theorem for a large class of higher order gravity theories. More precisely we show that, in the context of this class of theories, the set of spatially homogeneous cosmologies which can approach isotropy at late times is of measure zero in the space of all spatially homogeneous universe models. The proof is based on the transformation properties of the Raychaudhuri equation in higher order gravity theories and a recent extension of the Collins-Hawking theorem to scalar fields, Heusler's theorem. This result is used to present simplified proofs in higher order gravity of certain forms of the cosmic no-hair theorems in de Sitter and power-law inflation. We discuss the closed universe recollapse conjecture in a curvature-squared higher order gravity theory and give sufficient conditions for recollapse of the closed Friedmann model in the conformal frame.

Declaration

I declare that the contents of this thesis are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Ioannis Miritzis

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Time present and time past
Are both perhaps present in time future
And time future contained in time past

T. S. Elliot

Chapter 1

Introduction

The observable universe today seems to be remarkably homogeneous and isotropic on a very large scale and the Friedmann cosmology is a successful cosmological model capable of describing its large-scale properties. The most important cosmological discovery of recent decades has been the detection of the cosmic background radiation. A striking feature of the cosmic background radiation is a temperature isotropy over a wide range of angular scales on the sky. The remarkable uniformity of the cosmic background radiation indicates that at the end of the radiation-dominated period (some hundreds of thousands of years after the big bang) the universe was almost completely isotropic. One then has a difficulty in explaining why there should be such an isotropy in the universe for the following reason. The finite velocity of light divides the universe into causally decoherent regions. Roughly speaking, if the age of the universe is T, then regions moving away because of the expansion of the universe and separated by a distance greater than cT will not have enough time to communicate with each other. How did these separated regions come to be at the same temperature today to better than one part in ten thousand?

There are two currently popular responses to this so-called horizon problem. The first is that the universe has always been isotropic which means that the initial conditions were such that the universe was and has ever remained homogeneous and isotropic. This seems to be statistically quite improbable since the set of homogeneous and isotropic initial data is 'of measure zero' in the space of all initial data. The second response is that the universe came about in a less symmetric state and evolved through some dynamical mechanisms towards a FRW state. Soon after the discovery of the cosmic background radiation isotropy, Misner and others suggested that the universe could have started off in a 'chaotic state' with inhomogeneities and anisotropies of all kinds and that various dissipation processes could damp out nearly all of these, leaving only the very small amounts that we see today. This program was unable to show that the present state of the universe could be predicted independently of its initial conditions. Attempts to tackle to isotropization problem date at least since the pioneering work of Collins and Hawking [26] who formulated and proved the first isotropization theorems for certain classes of orthogonal Bianchi spacetimes.

Interest in a particular approach to the isotropization problem in cosmology renewed after the advent of inflation as a mechanism for solving the problem. Today it is believed that inflation is the most successful mechanism of isotropization and that the inflationary scenario answers simultaneously almost all problems of standard cosmology.

Soon after the invention of the inflationary scenario, interest focused on proving the so-called cosmic no-hair conjecture. This conjecture, roughly speaking, states that general cosmological initial data sets, when evolved through the gravitational field equations, are attracted (in a sense that can be made precise) by the de Sitter space of inflation. In other words, if this conjecture is true, inflation is a 'transient' attractor of such sets. These in turn quickly isotropize after the inflationary period and the inflationary regime could thus be regarded as 'natural' in view of its *prediction* of the (observed) large-scale homogeneity and isotropy of the universe.

No less interesting in modern cosmology is the recollapse problem, the question of whether or not closed universe models recollapse to a second all-encompassing singularity in their future. It has been demonstrated in the context of relativistic cosmology that only in very special cases, namely the closed FRW model [8], the closed spherically symmetric model [108, 24] and the orthogonal Bianchi IX with rather general matter fields [66] obeying appropriate energy conditions does one obtain a recollapsing universe. This problem is closely linked to that of the existence of constant mean curvature foliations in general relativity and is already quite involved in several 'simple' cases outside cosmology (see [87] for an interesting review).

Ever since the development of the singularity theorems of general relativity and their subsequent application to relativistic cosmology, a lot of interesting work has been focused (and is currently continuing) on the precise nature of the singularities in specific cosmological models, the best–known example being perhaps the Mixmaster universe, in particular the orthogonal and tilted Bianchi universes (for a review see [25]). More recently, Borde and Vilenkin [19] attempted to develop a series of singularity theorems applicable to inflationary cosmology in an effort to gain some understanding of the existence of an initial singularity in inflation.

From the above brief discussion we see that in recent years considerable progress has been made in the analysis of the three main unresolved issues of modern mathematical cosmology, namely the Singularity Problem, the Isotropization Problem and the Recollapse Problem. Although the final word on each one of these issues is still beyond our reach, we have a clear picture of the open problems in each of these areas and the complex interconnections that exist between different approaches to the above issues.

It is well known that the vacuum Einstein field equations can be derived from an action principle

$$S_E = \int d^4x\sqrt{-g}L_E \ , \tag{1.0.1}$$

where the Lagrangian L_E is just the Ricci scalar R

$$L_E = R. \tag{1.0.2}$$

If matter fields are included in the theory an appropriate L_{matter} term must be added to the Lagrangian (1.0.2).

Einstein was the first to modify his original theory in an attempt to obtain a static cosmological model. This modified theory can be derived from a Lagrangian

$$L = R - 2\Lambda, \tag{1.0.3}$$

where Λ is the cosmological constant. Since then there have been numerous attempts to generalize the action (1.0.1) by considering action functionals that contain curvature invariants of higher than first order in (1.0.3). These Lagrangians generally involved linear combinations of all possible second order invariants that can be formed from the Riemann, Ricci and scalar curvatures, namely

$$R^2, \quad R_{ab}R^{ab}, \quad R_{abcd}R^{abcd}, \quad \varepsilon^{iklm}R_{ikst}R^{st}_{lm}.$$

The reasons for considering higher order generalizations of the action (1.0.1) are multiple. Firstly, there is no *a priori* physical reason to restrict the gravitational Lagrangian to a linear function of R. Secondly it is hoped that higher order Lagrangians would create a first approximation to an as yet unknown theory of quantum gravity. For example a certain combination of the above second order invariants may have better renormalization properties than general relativity [98]. Thirdly one expects that, on approach to a spacetime singularity, curvature invariants of all orders ought to play an important dynamical role. Far from the singularity, when higher order corrections become negligible, one should recover general relativity. It is also hoped that these generalized theories of gravity might exhibit better behaviour near singularities.

In the following we critically review higher order gravity (HOG) theories, wherein the Lagrangian is an arbitrary smooth function of the scalar curvature, *ie*

$$L = f(R). \tag{1.0.4}$$

By varying L with respect to the metric tensor, $\mathbf{g}$, the action principle provides the vacuum field equations [6]

$$f' R_{ab} - \frac{1}{2} f g_{ab} - \nabla_a \nabla_b f' + g_{ab} \Box f' = 0, \tag{1.0.5}$$

where $\Box = g^{ab}\nabla_a\nabla_b$ and a prime ($'$) denotes differentiation with respect to R. These are fourth order equations, *ie* they contain fourth order derivatives of the metric. Therefore it is not surprising that very few solutions exist in the literature. For a discussion of cosmological solutions and stability issues see [12, 27, 32]. Among other difficulties related to the field equations (1.0.5) we mention the need for additional initial conditions in the formulation of the Cauchy problem besides the usual ones in general relativity.

Fortunately there is a method to overcome most of these problems. This is the conformal equivalence theorem proved by Barrow and Cotsakis [6]: Under a suitable conformal transformation, equations (1.0.5) reduce to the Einstein field equations with a scalar field as a matter source (for the technical details on conformally related metrics see Appendix D).

To see this we choose the conformal factor in (D.0.1) to be

$$\Omega^2 = f^{'}(R). \tag{1.0.6}$$

Granted the relation between the tensors R_{ab} and R in the spacetime $(M, \mathbf{g})$ to the corresponding ones $\widetilde{R}_{ab}$ and $\widetilde{R}$ in the spacetime $(M, \widetilde{\mathbf{g}})$, we may transform the field equations (1.0.5) to the new spacetime $(M, \widetilde{\mathbf{g}})$. With the introduction of a scalar field φ by

$$\varphi = \sqrt{\frac{3}{2}} \ln f^{'}(R), \tag{1.0.7}$$

the conformally transformed field equations become

$$\widetilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\widetilde{R} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}\tilde{g}_{ab}\left(\nabla_c\varphi\nabla^c\varphi\right) - \frac{1}{2}\tilde{g}_{ab}\left(f^{'}\right)^{-2}\left(Rf^{'} - f\right). \tag{1.0.8}$$

These are the Einstein equations for a scalar field source with potential

$$V = \frac{1}{2}\left(f^{'}\right)^{-2}\left(Rf^{'} - f\right). \tag{1.0.9}$$

The authors [6] state their result in D dimensions, but for our purposes a four-dimensional treatment is sufficient. If matter fields with energy-momentum tensor $T^m_{ab}(\mathbf{g})$ are present in the original spacetime $(M, \mathbf{g})$, the field equations become

$$f^{'}R_{ab} - \frac{1}{2}fg_{ab} - \nabla_a\nabla_b f^{'} + g_{ab}\Box f^{'} = T^m_{ab}(\mathbf{g}). \tag{1.0.10}$$

In the conformally related spacetime, $(M, \widetilde{\mathbf{g}})$, the corresponding Einstein equations become

$$\widetilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\widetilde{R} = T^m_{ab}(\widetilde{\mathbf{g}}) + T^{\varphi}_{ab}(\mathbf{g},\widetilde{\mathbf{g}}), \tag{1.0.11}$$

where $T^{\varphi}_{ab}(\mathbf{g},\widetilde{\mathbf{g}})$ is the right-hand side of (1.0.8) and $T^m_{ab}(\widetilde{\mathbf{g}})$ is the transformed stress-energy tensor.

The conformal equivalence theorem of Barrow and Cotsakis is a device to lower the order of the field equations to second and thus to obtain differential equations which are the usual Einstein equations plus the equation of motion of the scalar field. It is analogous to the Legendre transformation which reduces the second order Lagrange equations in classical mechanics to the first order Hamilton equations. However, several conceptual problems are raised. What does exactly the statement, 'the original higher order gravity theory is conformally equivalent to Einstein's equations plus a scalar field' imply? Does conformal equivalence imply physical equivalence between the two theories? If a certain property is proven in the conformal frame, is it also valid in the original spacetime? Which is the natural spacetime, the original one or the one conformally related to it? The last question is related to the issue of physical reality of the two metrics involved. In fact, in the conformal frame there are two possible candidates, the metrics $\mathbf{g}$ and $\widetilde{\mathbf{g}}$. A possible criterion of 'naturalness' of a metric would be the motion of matter: if matter moves on geodesics relative to one metric then this metric can be regarded as the physical one. For a discussion on the question of the physical reality of the two metrics involved and other interpretational issues see [28, 29] and references therein.

The conformal equivalence theorem allows certain rigorous results in general relativity to be transferred in HOG and therefore, to study the dynamical properties of higher order gravity theories by analyzing them in the conformal picture, that of Einstein's equations with a scalar field matter content. In the vacuum case this technique is very fruitful. However, in the presence of matter there are some limitations, mainly because the energy conditions imposed on the stress-energy tensor are *not* automatically conserved in the conformal frame. For example, if the stress-energy tensor satisfies the strong energy condition in $(M, \mathbf{g})$, the transformed tensor does not in general obey the same condition in $(M, \widetilde{\mathbf{g}})$. A sufficient (but not necessary) condition for the validity of the SEC in both spacetimes is that the stress-energy tensor be conformally invariant. By a *conformally invariant* stress-energy tensor T_{ab} we mean that $T_{ab} \to \widetilde{T}_{ab} = \Omega^w T_{ab}$, where the number $w \geq 0$ is the *conformal weight* of the tensor. This will be the case if T_{ab} is derived from a conformally invariant action with respect to the metric.

Higher derivative quantum corrections to the gravitational action of classical general relativity are generally expected to play a significant role at very high energies where a quantum gravitational field will presumably dominate. It is not unreasonable to consider classical cosmology in theories coming out of such nonlinear gravitational Lagrangians and in fact, one expects that there exist close links between properties of such 'higher derivative cosmologies' and those of general relativistic cosmology. It is obvious that the resolution of aspects of the singularity, isotropization and recollapse problems is of paramount importance also in this extended framework.

In this thesis we examine structural and cosmological issues in gravity theories obtained from a Lagrangian $L = f(R)$. The plan of this thesis is the following:

- Chapter Two: In this chapter we begin by critical review of the Palatini variation in general relativity which outlines the method. Next we derive the field equations via the Palatini variational principle for general gravitational Lagrangians which are functions of all possible curvature invariants. We find that the Palatini method leads to certain restrictions on the form of the Lagrangians and, if matter fields are included, to severe inconsistencies. We carefully formulate an extension of the Palatini method, the so-called constrained Palatini variation, and prove a general result of equivalence of this generalized variational procedure to the usual Hilbert method for most of the higher order Lagrangians. We apply this theorem to $f(R)$ theories in the extended framework of Weyl geometries and prove a non-trivial generalization of the conformal equivalence theorem. The forms of the scalar field potential and conformal factor are formally identical to the usual ones. However, new issues arise relative to the stress tensor of matter fields in this picture. In particular, previous results valid in the Riemannian framework appear naturally as special cases of this general result in the limit when Weyl geometries tend to Riemannian ones.

- Chapter Three: After a brief review of the inflationary scenario, we carefully state the cosmic no-hair conjecture and critically discuss its limitations. Next we prove the central result, namely the cosmic no-hair conjecture for all orthogonal Bianchi cosmologies with matter in the $R+\beta R^2+L_{matter}$ theory. The proof is based on the conformally equivalent Einstein field equations with the scalar field that has the usual self-interacting potential in the presence of the conformally related matter fields. We show, in particular, that the Bianchi IX universe asymptotically approaches de Sitter space provided that initially the scalar three-curvature does not exceed the potential of the scalar field associated with the conformal transformation. We also show that the time needed for the scalar field to reach the minimum of the potential is much larger than the isotropization time.

- Chapter Four: In this Chapter we present a generalization of the Collins-Hawking theorem for a large class of higher order gravity theories. More precisely we show that in the context of this class of theories, the set of spatially homogeneous cosmologies which can approach isotropy at late times is of measure zero in the space of all spatially homogeneous universe models. This involves a non-trivial argument based on a treatment of the Raychaudhuri equation in HOG theories and a recent extension of the Collins-Hawking theorem to scalar fields [53]. The result is also used to prove in HOG theories certain known forms of cosmic no-hair theorems in de Sitter and power-law inflation in general relativity.

- Chapter Five: We discuss the closed universe recollapse conjecture in a curvature-squared higher order gravity theory. In particular we give sufficient conditions for recollapse of the closed Friedmann model in the conformal frame.

- Chapter Six: We present our conclusions and suggest some paths of future research.

For the convenience of the reader we include four appendices with the necessary material and geometric notions that are needed for the development of this thesis.

Notation and conventions

In this thesis we follow the sign conventions of Misner, Thorne and Wheeler (MTW [78]). In particular we use the metric signature $(-,+,+,+)$ and define the Riemann tensor by

$$R\,(X,Y)\,Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

so that

$$\nabla_c \nabla_d Z^a - \nabla_d \nabla_c Z^a = R^a{}_{bcd} Z^b.$$

The Ricci tensor is defined as the one-three contraction of the Riemann tensor so that

$$R_{bd} = R^a{}_{bad}.$$

The Einstein tensor is defined as

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R,$$

where g_{ab} is the metric tensor and R is the scalar curvature tensor. The Einstein field equations are

$$G_{ab} = T_{ab}.$$

Throughout this work, we use units where $c = 8\pi G = 1$.

We also employ the *abstract index notation* discussed in Wald [101]. Thus Latin indices of a tensor denote the type of the tensor (they are part of the notation for the tensor itself). Greek indices on a tensor represent its components in a given frame. In the cases where purely spatial tensor components occur, the range of the indices is explicitly denoted. An exception to the above convention is Chapter 2, where all indices are component indices.

∇_a is the symbol for the covariant derivative operator. We occasionally use a semicolon $(;)$ to denote covariant differentiation. The symbol ∂_a stands for the ordinary derivative operator.

Here are some abbreviations most frequently used in the text.

HOG	Higher order gravity
HE	Hawking and Ellis
FRW	Friedmann-Robertson-Walker
PV	Palatini variation
CPV	Constrained Palatini variation
HDCs	Higher Derivative Cosmologies
CBR	Cosmic background radiation
GR	General relativity
SEC	Strong energy condition
WEC	Weak energy condition
TCC	Timelike convergence condition
DEC	Dominant energy condition
CNHC	Cosmic no-hair conjecture
CNHT	Cosmic no-hair theorem
PPC	Positive pressure criterion

The stress-energy-momentum tensor is usually written as stress-energy tensor.

Part I

Variational issues in higher order gravity theories

Chapter 2

Constrained variations and nonlinear Lagrangians

In this Chapter we generalize the constrained Palatini variation (CPV) in order to include arbitrary symmetric connections in gravity theories derived from nonlinear Lagrangians. In Section 2.1 we review the two variational principles namely the Hilbert and Palatini variations in general relativity. In Section 2.2 we apply the Palatini variation to nonlinear gravitational Lagrangians. In particular we extend Buchdahl's work [23] to $f\left(Ric^2\right)$ Lagrangians and give an alternative approach to the recent analysis of Borowiec *et al* [21]. A critique on the Palatini variation serves as a motivation for Section 2.3 where we present the constrained Palatini variation and prove a general theorem to the effect that the CPV field equations obtained from the most general gravitational Lagrangian with general couplings to matter fields are identical to those derived by the Hilbert variation. We apply this theorem to quadratic Lagrangians and correct previous results. In Section 2.4 we derive the field equations for an $f\left(R\right)$ Lagrangian in Weyl geometry, study the conformal structure of these theories and extend previous results [6] valid in Riemannian geometry. Furthermore our analysis shows that the Palatini variation can be recovered as a special case of the CPV. In the last Section we discuss our results and compare them with those in previous works.

2.1 Variational principles in General Relativity

Soon after the discovery of general relativity, Einstein, Weyl, Eddington and Cartan among others, were searching for a more general theory that would unify on purely geometrical grounds gravitational and electromagnetic phenomena. Several ideas were introduced and developed at that epoch concerning the structure of the underlined spacetime manifold, such as semi-riemannian structures in dimension greater than four, non-symmetric metric tensor, connection non-compatible with the metric, no imposition of a metric from the beginning but such that a metric structure could result as a byproduct (for example, as the symmetric part of the Ricci tensor R_{ab}, or the symmetric part of the functional derivative of a gravitational Lagrangian with respect to the Ricci tensor). The variational principle proved to be an indispensable tool in the derivation of these theories. In fact, since Hilbert succeeded in deriving the Einstein field equations in vacuum by varying the action $\int R\sqrt{-g}$ with respect to the metric, it became clear that variational principles should play an important role in setting up a theory of gravity.

The standard variational principle (Hilbert variation) that leads to general relativity is a metric variation, in the sense that the gravitational action is varied only with respect to the metric tensor. It is assumed from the beginning that the underling manifold is a four-dimensional Lorentz manifold $(M, \mathbf{g}, \nabla)$ with metric $\mathbf{g}$ and the Levi-Civita connection ∇. However, in the search of a possible generalization of GR one could consider a Lorentz manifold with an arbitrary connection ∇ which is totally unrelated to the metric, *ie* $\nabla \mathbf{g} \neq 0$. A motivation for such a generalization was inspired by the work of Weyl [103] wherein for the first time the notion of distance appeared as independent of the notion of parallelism (other possible motivations may be found in [78]). Therefore, as an alternative procedure, one could consider variation of the action with respect to both the metric components g_{ab} and the connection coefficients Γ^a_{bc} without imposing from the beginning that Γ^a_{bc} be the usual Christoffel symbols. This method was used for the first time by Einstein in 1925 [40]. By historical accident[1] the method is attributed to Palatini, although the latter was not responsible for that. In the current literature the variational principle, where the metric and the connection are considered as independent variables, is referred to as the *Palatini variation.* Even Einstein in his late years used to refer to Palatini without mentioning his own previous work, probably in order to follow the by then accepted custom.

2.1.1 Hilbert variation

The simplest gravitational Lagrangian is the Einstein-Hilbert Lagrangian (1.0.2)

$$L = R,$$

($R = g^{ab}R_{ab}$ is the scalar curvature). The corresponding action is

$$S = \int L\sqrt{-g}d^4x,$$

where the integral is taken over a compact region U of the spacetime $(M, \mathbf{g}, \nabla)$. Here ∇ is the Levi-Civita connection of the metric, *ie* $\nabla\mathbf{g} = 0$. We assume that the metric components and their derivatives are held constant on the boundary of U. The Ricci tensor is expressed in terms of the connection coefficients Γ^a_{bc} and their first derivatives and Γ^a_{bc} are related to the metric by the usual relations, *ie* they are the Christoffel symbols. Hence the Lagrangian density $\mathfrak{L} := L\sqrt{-g}$ can be written as

$$\mathfrak{L} = \mathfrak{L}\left(g, \partial g, \partial^2 g\right).$$

Performing an arbitrary variation δg_{ab} to the metric, one could calculate the variation δS of the action and, setting $\delta S = 0$, one would obtain the field equations. This is Hilbert's original prescription for the derivation of the Einstein equations in vacuum. However, we could proceed in the following way. Consider a manifold M (not necessarily semi-Riemannian) equipped with two connections ∇ and $\widetilde{\nabla}$. As mentioned in the previous chapter, the corresponding connection coefficients Γ and $\widetilde{\Gamma}$ differ by a tensor field, *ie*

$$\delta\Gamma^c_{ab} = \widetilde{\Gamma}^c_{ab} - \Gamma^c_{ab}$$

[1]For an interesting historical restoration of this affair see Ferraris *et al* [42].

is a tensor field. Different connections lead to different curvature tensors, but the difference between $\widetilde{R}^a{}_{bcd}$ and $R^a{}_{bcd}$ is given by a simple formula due to Palatini. This is the *Palatini equation*

$$\delta R^a{}_{bcd} = \nabla_c \left(\delta\Gamma^a_{bd}\right) - \nabla_d \left(\delta\Gamma^a_{bc}\right), \tag{2.1.1}$$

which can be easily derived in a locally geodesic coordinate system (see for example [56]). Contraction yields

$$\delta R_{ab} = \nabla_m \left(\delta\Gamma^m_{ab}\right) - \nabla_b \left(\delta\Gamma^m_{am}\right). \tag{2.1.2}$$

This identity allows the evaluation of the variation of the Ricci tensor due to an arbitrary variation of the connection. In particular it applies in the case where the variation of the Levi-Civita connection is due to an arbitrary variation of the metric.

We are now in the position to calculate the variation of the action as[2]

$$0 = \delta \int wL = \int \big(wR_{ab}\delta g^{ab} + w\delta R_{ab} g^{ab} + R\delta w\big). \tag{2.1.3}$$

Using the Palatini identity (2.1.2) the middle term in the right-hand side of (2.1.3) can be written as

$$\int wg^{ab} \left(\nabla_m \left(\delta\Gamma^m_{ab}\right) - \nabla_b \left(\delta\Gamma^m_{am}\right)\right)$$

and after integrating by parts it becomes an integral of a pure divergence since the covariant derivative of both g_{ab} and w is zero. By Gauss' theorem this integral vanishes since we have assumed that the metric and its derivatives vanish on the boundary of the region of integration. The variation of w given by $\delta w = -\frac{1}{2} w g_{ab} \delta g^{ab}$ and equation (2.1.3) finally becomes

$$\int w \left(R_{ab} - \frac{1}{2} R g_{ab}\right) \delta g^{ab} = 0.$$

Since the variation δg^{ab} and the region of integration U are arbitrary, we conclude that the last equation implies the Einstein field equations in vacuum.

The full field equations are obtained if we add to the gravitational Lagrangian (1.0.2) an appropriate Lagrangian L_m for the matter fields.

[2] In the following we omit the symbol d^4x under the integral sign and set $w := \sqrt{-g}$. Gothic characters denote tensor densities, for example $\mathfrak{g}_{ab} := w g_{ab}$.

The matter Lagrangian, depending primarily on the field variables which we collectively call ψ, is a generalization of its special relativistic form. This generalization is achieved via the principle of equivalence (and the undefined principle of 'simplicity') according to the scheme $\eta_{ab} \to g_{ab}$ and $\partial \to \nabla$, the order of the two steps being irrelevant as long as the connection is the Levi-Civita one.[3] Therefore, L_m is a function of the matter fields present and of the metric and its derivatives. Varying the total action with respect to g_{ab} we obtain

$$\delta \int w \left(L + L_m\right) =: \delta \int w \left(-G^{ab} + T^{ab}\right) \delta g_{ab}. \tag{2.1.4}$$

Equation (2.1.4) defines the stress-energy tensor T_{ab}. (The minus in front of the contravariant Einstein tensor comes out from the identity $g^{ab}\delta g_{ab} = -g_{ab}\delta g^{ab}$). Setting the variation of the total action equal to zero we obtain the Einstein field equations.

We end up this paragraph with an important differential identity implied by the metric variational principle [39]. Consider an arbitrary Lagrangian which is assumed to be a function of the metric and its (possibly higher order) derivatives as well as of some other fields ψ, that is

$$L = L\left(g, \partial g, \partial^2 g, ..., \psi\right). \tag{2.1.5}$$

The variation of the action

$$\delta \int \mathfrak{L} =: \int \mathfrak{A}^{ab} \delta g_{ab} \tag{2.1.6}$$

defines the functional derivative $\mathfrak{A}^{ab}$ (or the Euler-Lagrange derivative) of the Lagrangian density $\mathfrak{L}$. The field equations are then

$$\mathfrak{A}^{ab} = 0.$$

A specific class of variations of the metric is that induced by diffeomorphisms $f : M \to M$. With any diffeomorphism of spacetime we may associate a coordinate transformation (see, for instance, HE [50]). Since the manifolds $(M, \mathbf{g})$ and $(M, f^*\mathbf{g})$ are physically equivalent, the action

[3] However, for arbitrary connections there is an ambiguity because the operation of lowering and raising indexes does no longer commute with the covariant derivative operator.

functional does not change under the diffeomorphism f. In particular, the action remains unchanged under an infinitesimal coordinate transformation. It is well known that, for such variations, δg_{ab} is associated with the Lie derivative $Ł_X g_{ab}$ of the metric with respect to the infinitesimal generator X of the diffeomorphism, *ie* $\delta g_{ab} = \nabla_a X_b + \nabla_b X_a$. Since by definition (2.1.6) $\mathfrak{A}^{\mathfrak{ab}}$ is symmetric, the variational principle yields

$$2\int \mathfrak{A}^{ab} \left(\nabla_a X_b\right) = 0$$

for all vector fields X vanishing on the boundary of integration. Partially integrating the last equation and discarding the divergence term we obtain

$$\int \left(\nabla_a \mathfrak{A}^{ab}\right) X_b = 0,$$

which implies the promising identity

$$\nabla_a \mathfrak{A}^{ab} = 0. \tag{2.1.7}$$

We shall call (2.1.7) the *generalized Bianchi identities* since in the case of the Einstein-Hilbert Lagrangian (1.0.2) they take the form of the contracted Bianchi identities, namely

$$\nabla_a G^{ab} = 0.$$

2.1.2 Palatini variation

As mentioned in the introduction of Section 2.1, one could start with a (semi-)Riemannian manifold $(M, \mathbf{g}, \nabla)$ where ∇ is an arbitrary symmetric connection, *ie* $\nabla \mathbf{g} \neq 0$, and $\nabla_X Y - \nabla_Y X = [X, Y]$ for all vector fields X and Y defined on M. Hence, the connection coefficients are considered as functions independent of the metric components. In the case of general relativity, the gravitational Lagrangian $L = R = g^{ab} R_{ab}$ is regarded as a function of the 10 metric components g_{ab} and the 40 connection coefficients Γ^a_{bc}. Therefore, the corresponding variational principle (the Palatini variation) consists of an independent or separate variation of the metric and the connection. It turns out that this method is technically much simpler than the metric variation discussed above. In fact, if we carry out a variation with respect to g^{ab} only, we find $\delta R_{ab} = 0$ since the Ricci tensor depends only on the connection. Hence, (compare to (2.1.3))

$$0 = \delta \int \left(w R_{ab} \delta g^{ab} + R \delta w \right) = \delta \int w \left(R_{ab} - \frac{1}{2} R g_{ab} \right) \delta g^{ab},$$

which immediately gives the Einstein equations in vacuum. Variation with respect to Γ^a_{bc} yields

$$0 = \int w \delta R_{ab} g^{ab} = \int w g^{ab} \left[\nabla_c \left(\delta \Gamma^c_{ab} \right) - \nabla_b \left(\delta \Gamma^c_{ac} \right) \right],$$

where the Palatini identity (2.1.2) was used. Integrating by parts we have

$$0 = \int \left[\nabla_b \left(\mathfrak{g}^{ab} \delta \Gamma^c_{ac} \right) - \nabla_c \left(\mathfrak{g}^{ab} \delta \Gamma^c_{ab} \right) \right] = \int \left(\delta^b_c \nabla_d \mathfrak{g}^{ad} - \nabla_c \mathfrak{g}^{ab} \right) \delta \Gamma^c_{ab}$$

and, since $\delta \Gamma^c_{ab}$ are arbitrary, it follows that the symmetric part in a and b of the expression in brackets vanishes, *ie*

$$\delta^b_c \nabla_d \mathfrak{g}^{ad} + \delta^a_c \nabla_d \mathfrak{g}^{bd} - 2 \nabla_c \mathfrak{g}^{ab} = 0, \qquad (2.1.8)$$

which by standard arguments implies that the covariant derivative of $\mathfrak{g}_{ab}$ (and of g_{ab}) vanishes, *ie* $\nabla_c g_{ab} = 0$. It follows that the Γ^c_{ab} are necessarily the Christoffel symbols,

$$\Gamma^c_{ab} = \frac{1}{2} g^{cm} \left(g_{am,b} + g_{bm,a} - g_{ab,m} \right).$$

Thus we arrive at the well-known result that variation with respect to the metric produces the vacuum Einstein equations and variation with respect to the connection reveals that the connection is necessarily the Levi-Civita connection. That is why the Palatini variation is usually referred to in the literature as equivalent to the Hilbert variation. This is erroneous since, as we shall see, the fact that the two variational principles, if applied to the Lagrangian $L = R$, produce the same theory is a mere coincidence. In the presence of matter fields described by an arbitrary matter Lagrangian the two variations are in general inequivalent. When the matter Lagrangian does not depend explicitly on the connection, the two variations are equivalent. This is the case for most known forms of matter fields: For example, the Lagrangian of a massive scalar field contains only ordinary derivatives of the field and in the case of electromagnetism the derivatives of the field A never appear alone, but always in the combination $\nabla_a A_b - \nabla_b A_a$ making covariant derivatives equivalent to ordinary derivatives. In the case of generalized Yang-Mills field also, the Palatini method is equivalent to the Hilbert variation. However, the ultimate Lagrangian of the real world is not yet known. One can construct examples where the two methods are inequivalent. In some cases, as is the case of Einstein-Dirac fields, the equivalence is restored by adding an appropriate term to the Lagrangian. Nevertheless, the main drawback of the Palatini variation is not its inequivalence to the Hilbert variation, but stems from the fact that it leads to certain inconsistencies in the field equations obtained from a general Lagrangian [58, 52].

To see this consider an arbitrary matter Lagrangian $L_m(g, \psi, \nabla\psi)$ so that the total action is

$$S = \int \left[\mathfrak{R}(g, \nabla) + \mathfrak{L}_m(g, \psi, \nabla\psi)\right],$$

where we have emphasized the explicit dependence of the scalar curvature $R = g^{ab}R_{ab}$ on both the metric and the connection. For an arbitrary (but symmetric) connection the Palatini method gives the following pair of equations:

$$G_{ab} = T_{ab} := -\frac{1}{w}\frac{\delta \mathfrak{L}_m}{\delta g^{ab}}, \tag{2.1.9}$$

$$\delta^b_c \nabla_d \mathfrak{g}^{ad} + \delta^a_c \nabla_d \mathfrak{g}^{bd} - 2\nabla_c \mathfrak{g}^{ab} = 2\frac{1}{w}\frac{\delta \mathfrak{L}_m}{\delta \Gamma^c_{ab}}. \tag{2.1.10}$$

(Compare (2.1.10) to (2.1.8)). These equations are inconsistent in general[4] and they are *not* equivalent to the full Einstein equations obtained via the metric variation unless the matter Lagrangian does not depend explicitly on the connection, *ie* $\delta\mathfrak{L}_m/\delta\Gamma^c_{ab} = 0$.

It is interesting to note that, in two dimensions in vacuum, the Palatini method leaves the connection undetermined. In fact it is not difficult to show from (2.1.8) that for any dimension D,

$$\Gamma^c_{ab} = \left\{{}^{\,c}_{ab}\right\} + \frac{1}{2}\left(\delta^c_a Q_b + \delta^c_b Q_a - g_{ab}Q^c\right) \tag{2.1.11}$$

[4]This inconsistency is due to the fact that the geometric parts of these equations are projectively invariant whereas the sources are not in general (see [52]).

with $Q_a := -\nabla_a \ln w = -\partial_a \ln w + \Gamma_a$, and $\Gamma_a := \Gamma^b_{ab}$ The trace of (2.1.11) is

$$\left(1 - \frac{D}{2}\right)\left(\partial_a \ln \sqrt{-g} - \Gamma_a\right) = 0.$$

Hence the Γ_a part of the connection is undetermined in two dimensions [38].

2.2 Palatini variation for general Lagrangians

Nonlinear Lagrangians in the context of alternative variational methods were first considered by Weyl and Eddington [103, 39]. Later, in attempts to obtain second order differential equations different from Einstein's equations, Stephenson [96] and Higgs [54] applied the Palatini variation to the quadratic Lagrangians R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$. Yang [106] investigated a theory based on the Lagrangian $R_{abcd}R^{abcd}$, by analogy with the Yang–Mills Lagrangian. However, as Buchdahl [22] pointed out, there was a conceptual mistake in the variational method used by Stephenson and Yang, for they considered independent variations of the metric and the connection and imposed the metricity condition, *ie* the connection coefficients equal to the Christoffel symbols, *after* the variation. Purely Palatini variations for quadratic Lagrangians without imposing the metricity condition were considered by Buchdahl [23] who showed using specific examples that the PV is not a reliable method in general. Van den Bergh [15] arrived at a similar conclusion in the context of general scalar-tensor theories. The $R+\alpha R^2$ theory including matter has been investigated by Shahid-Saless[91] and was generalized to the $f(R)$ case by Hamity and Barraco [49]. These authors also studied conservation laws and the weak field limit of the resulting equations. More recently, Ferraris *et al* [43] have shown that the Palatini variation of $f(R)$ vacuum Lagrangians leads to a series of Einstein spaces with cosmological constants determined by the explicit form of the function f . Similar results were obtained in the case of $f(Ric^2)$ theories by Borowiec *et al* [21].

As expected, for more general nonlinear Lagrangians in a four-dimensional spacetime of the type $f(R)$, $f\left((Ricci)^2\right)$, $f\left((Riemann)^2\right)$, where f is an arbitrary smooth function, the field equations obtained from a Palatini variation are of second order while the corresponding ones obtained via the usual metric variation are of fourth order. This result (see below for explicit derivations) sounds very interesting since it could perhaps lead to an alternative way to 'cast' the field equations of these theories in a more tractable reduced form than the one we usually use for this purpose namely, the conformal equivalence theorem [6]. In this way certain interpretational issues related to the question of the physicality of the two metrics [28, 29] associated with the conformal transformation would be avoided. Unfortunately, as we show below, there are other difficulties that appear when one follows the method which are more serious than those encountered in the conformal transformation method. The net result is that, viewed as an alternative to, for instance, reducing the complexity of the field equations, the Palatini variation is not a reliable method (see, however, Section 2.3).

2.2.1 $f(R)$ Lagrangians

We begin with a Lagrangian that is a smooth function of the scalar curvature R,

$$L = f\left(R\right). \tag{2.2.1}$$

The Ricci tensor is build up of only the connection Γ and its derivatives, *ie* is independent of the metric. We vary the action

$$S = \int wf\left(R\right) \tag{2.2.2}$$

with respect to the metric tensor and set the variation equal to zero:

$$0 = \int \left[f\left(R\right)\delta w + w\delta f\left(R\right)\right] = \int \left[-\frac{1}{2}f\left(R\right)g_{ab} + f^{'}\left(R\right)R_{(ab)}\right]w\delta g^{ab}, \tag{2.2.3}$$

where the identity,

$$\delta w = \frac{1}{2}wg^{ab}\delta g_{ab} = -\frac{1}{2}wg_{ab}\delta g^{ab},$$

was used and $f^{'}$ is an abbreviation for $f^{'}\left(R\right) := df/dR$. Since the δg^{ab} are arbitrary, (2.2.3) implies

$$f^{'}R_{(ab)} - \frac{1}{2}fg_{ab} = 0. \tag{2.2.4}$$

We now vary the action (2.2.2) with respect to the connection Γ:

$$0 = \int w f'(R)\, g^{ab} \delta R_{ab} = \int w f' g^{ab} \left[\nabla_m \left(\delta \Gamma^m_{ab} \right) - \nabla_b \left(\delta \Gamma^m_{am} \right) \right],$$

where the Palatini identity (2.1.2) was used. Partial integration of the right-hand side of the last equation yields

$$\begin{aligned} 0 &= \int \left[-\nabla_m \left(w f' g^{ab} \right) \delta \Gamma^m_{ab} + \nabla_b \left(w f' g^{ab} \right) \delta \Gamma^m_{am} \right] + \int \text{total divergence} \\ &= \int \left[\delta^b_c \nabla_m \left(w f' g^{am} \right) - \nabla_c \left(w f' g^{ab} \right) \right] \delta \Gamma^c_{ab}. \end{aligned}$$

Since the $\delta\Gamma^c_{ab}$ are independent, it follows that the symmetric part (in a and b) of the term in brackets which is multiplied by $\delta\Gamma^c_{ab}$ vanishes, *ie*

$$\delta^b_c \nabla_m \left(w f' g^{am} \right) + \delta^a_c \nabla_m \left(w f' g^{bm} \right) - 2 \nabla_c \left(w f' g^{ab} \right) = 0. \qquad (2.2.5)$$

Contraction on b and c gives $\nabla_m \left(w f' g^{am} \right) = 0$ which, when inserted back to (2.2.5), yields finally

$$\nabla_a \left(w f' g^{bc} \right) = 0. \qquad (2.2.6)$$

It is evident that the field equations (2.2.4) and (2.2.6) derived from the Lagrangian (2.2.1) differ from the corresponding field equations obtained from the same Lagrangian with a Hilbert variation. To see what these equations imply, let us first consider in detail (2.2.6) which is usually called the $\Gamma-$ equation. Expanding (2.2.6) we have

$$f' g^{bc} \nabla_a w + w f' \nabla_a g^{bc} + w f'' g^{bc} \nabla_a R = 0.$$

With the use of $\nabla_a w = \partial_a w - \Gamma_a w$ and $\nabla_a R = \partial_a R$ the last equation can be written as

$$\left(\partial_a \ln w + \left(\ln f' \right)' \partial_a R - \Gamma_a \right) g_{bc} - \partial_a g_{bc} + \Gamma^m_{ba} g_{mc} + \Gamma^m_{ca} g_{mb} = 0. \qquad (2.2.7)$$

Contracting with g^{bc} we see that Γ_a is a gradient:

$$\Gamma_a = \partial_a \ln w + 2 \left(\ln f' \right)' \partial_a R. \qquad (2.2.8)$$

Had we begun with an asymmetric Ricci tensor, we would end up with a symmetric Ricci tensor since $R_{[ab]} = \partial_{[a} \Gamma_{b]}$. Inserting the value of Γ_a in (2.2.7) one can show that

$$\partial_a \left(f' g_{bc} \right) = \Gamma^m_{ba} f' g_{mc} + \Gamma^m_{ca} f' g_{mb}. \qquad (2.2.9)$$

This suggests that if we define a conformally related metric with conformal factor f', we could 'make' the connection Levi-Civita. Indeed, if we set

$$\widetilde{g}_{ab} := f^{'} g_{ab}, \tag{2.2.10}$$

(2.2.9) becomes

$$\partial_a \widetilde{g}_{bc} = \Gamma^m_{ba} \widetilde{g}_{mc} + \Gamma^m_{ca} \widetilde{g}_{mb} \tag{2.2.11}$$

which implies that the covariant derivative of the new metric $\widetilde{\mathbf{g}}$ with respect to the connection Γ vanishes, *ie* the connection Γ is the Levi-Civita connection for the metric $\widetilde{\mathbf{g}}$.

On the other hand the analysis of the field equation (2.2.4) is more straightforward. By taking the trace of (2.2.4) we find

$$f^{'}(R)\, R = 2f(R)\,. \tag{2.2.12}$$

This equation is identically satisfied by the function

$$f(R) = R^2, \tag{2.2.13}$$

apart from a constant rescaling factor. Accordingly the field equation (2.2.4) is

$$R_{ab} - \frac{1}{4} R g_{ab} = 0, \tag{2.2.14}$$

provided that $f^{'}(R) \neq 0$. Observe that the scalar curvature in the original frame is undetermined because (2.2.14) is traceless, which is another peculiarity of the Palatini method. Moreover, since R_{ab} is constructed only from the connection and its derivatives, it remains unchanged under the transformation $g_{ab} \rightarrow f^{'} g_{ab}$, *ie* $\widetilde{R}_{ab} = R_{ab}$. Hence, the field equation (2.2.14) becomes

$$\widetilde{R}_{ab} - \frac{1}{4} \widetilde{g}_{ab} = 0. \tag{2.2.15}$$

The conformally equivalent field equation (2.2.15) implies that the underlined manifold must be an Einstein space[5] with unit scalar curvature $\widetilde{R} = \widetilde{g}^{ab} \widetilde{R}_{ab}$.

However, the quadratic solution (2.2.13) to the trace equation (2.2.12) is not the only possibility. Given an arbitrary differentiable function f, (2.2.12) can be regarded as an algebraic equation to be solved for R.

[5]An Einstein space is defined by the property that $Ric = c\,\mathbf{g}$, where c is a constant.

Denoting the resulting roots by $\rho_1, \rho_2, \ldots$ one obtains a whole series of Einstein spaces, each having a constant scalar curvature. This situation was analyzed by Ferraris *et al* [43] (who generalized the case $a+bR+R^2$ studied previously by Buchdahl [23]).

However, if it happens for some root ρ_i to be $f'(\rho_i) = 0$, then the trace equation (2.2.12) implies that $f(\rho_i) = 0$ also. In that case the field equations leave both the metric and the connection completely undetermined.

2.2.2 $f(R_{(ab)}R^{ab})$ Lagrangians

We consider a Lagrangian

$$L = f(r), \tag{2.2.16}$$

where $r = Q_{ab}R^{ab}$ and Q_{ab} is the symmetric part[6] of the Ricci tensor. Remembering that the Ricci tensor depends only on the connection, by varying the corresponding action with respect to the metric we obtain

$$0 = \delta \int wf(r) = \int \left[-\frac{1}{2} wfg_{ab}\delta g^{ab} + 2wf' g^{am}(Q_{ab}Q_{mn})\delta g^{bn} \right]$$

or, renaming the indices,

$$0 = \int \left[-\frac{1}{2} fg_{ab} + 2f' g^{mn}(Q_{ma}Q_{nb}) \right] w\delta g^{ab}.$$

The variation of the metric being arbitrary, the last equation implies

$$f' Q_{ac}Q^{c}{}_{b} - \frac{1}{4} fg_{ab} = 0. \tag{2.2.17}$$

We now vary the action with respect to the connection. As in the previous Section we express the variation of the Ricci tensor via the Palatini identity (2.1.2) and after the integration by parts we discard the divergence terms:

$$\begin{aligned}
0 &= 2\int wf' Q^{ab}\delta Q_{ab} = 2\int wf' Q^{ab}\left[\nabla_m(\delta\Gamma^m_{ab}) - \nabla_b(\delta\Gamma^m_{am})\right] \\
&= 2\int \left[-\nabla_m\left(wf' Q^{ab}\right)\delta\Gamma^m_{ab} + \nabla_b\left(wf' Q^{ab}\right)\delta\Gamma^m_{am} \right] \\
&= 2\int \left[\delta^b_c \nabla_m\left(wf' Q^{am}\right) - \nabla_c\left(wf' Q^{ab}\right) \right]\delta\Gamma^c_{ab}.
\end{aligned}$$

[6] The field equations derived from the Lagrangian $R_{[ab]}R^{ab}$ by the Palatini variation impose only four conditions upon the forty connection coefficients and leave the metric components entirely undetermined [23].

It follows that

$$\delta^b_c \nabla_m \left(w f^{'} Q^{am}\right) + \delta^a_c \nabla_m \left(w f^{'} Q^{bm}\right) - 2\nabla_c \left(w f^{'} Q^{ab}\right) = 0. \quad (2.2.18)$$

Contracting the last equation on b and c we obtain

$$\nabla_m \left(w f^{'} Q^{am}\right) = 0. \quad (2.2.19)$$

Inserting (2.2.19) into (2.2.18) we finally obtain

$$\nabla_a \left(w f^{'} Q^{bc}\right) = 0. \quad (2.2.20)$$

To see what the $\Gamma-$ equation (2.2.20) implies we first expand

$$\begin{aligned} &\left(\partial_a \ln w - \Gamma_a\right) Q^{bc} + \left(\ln f^{'}\right)^{'} Q^{bc} \left(\partial_a Q^{mn} Q_{mn} + Q^{mn} \partial_a Q_{mn}\right) \\ &+\partial_a Q^{bc} + \Gamma^b_{ma} Q^{mc} + \Gamma^c_{ma} Q^{cm} = 0. \end{aligned} \quad (2.2.21)$$

Assuming that $rank\, Q^{ab} = 4$ and following Buchdahl [23], we define the 'reciprocal' tensor P_{ab} :

$$P_{am} Q^{mb} = \delta^b_a \quad (2.2.22)$$

and set

$$p := \det Q_{ab} = w^4 \det Q^{ab}. \quad (2.2.23)$$

Then, with the usual identity $\partial_a \left(\det Q^{ab}\right) / \det Q^{ab} = P_{mn} \partial_a Q^{mn}$, we find

$$P_{mn} \partial_a Q^{mn} = \partial_a \ln \left(w^{-4} p\right) = -4 \partial_a \ln w + \partial_a \ln p. \quad (2.2.24)$$

We can solve for Γ_a equation (2.2.21) by multiplying it with P_{bc} and using (2.2.24). The result is

$$\Gamma_a = \frac{1}{2} \partial_a \ln p + 2 \left(\ln f^{'}\right)^{'} \partial_a \left(Q_{mn} Q^{mn}\right). \quad (2.2.25)$$

This equation shows that Γ_a is a gradient, hence the Ricci tensor is symmetric and we can replace Q_{ab} by R_{ab} everywhere.

If we multiply (2.2.21) with $P_{bm} P_{cn}$, we find that

$$\left[\partial_a \ln w - \frac{1}{2} \partial_a \ln p - \left(\ln f^{'}\right)^{'} \partial_a \left(R_{bc} R^{bc}\right)\right] P_{mn} - \partial_a P_{mn} + \Gamma^b_{na} P_{bm} + \Gamma^c_{ma} P_{cn} = 0, \quad (2.2.26)$$

where the value of Γ_a was substituted from (2.2.25). We observe now that, if we define

$$\widetilde{g}_{ab} := \frac{f' p^{1/2}}{w} P_{ab} \tag{2.2.27}$$

and

$$\widetilde{g}^{ab} = \frac{w}{f' p^{1/2}} R^{ab} \tag{2.2.28}$$

so that $\widetilde{g}_{ab}\widetilde{g}^{bc} = \delta^c_a$, then (2.2.26) becomes

$$\Gamma^b_{na}\widetilde{g}_{bm} + \Gamma^c_{ma}\widetilde{g}_{cn} = \partial_a \widetilde{g}_{mn}. \tag{2.2.29}$$

As a consequence the covariant derivative of $\widetilde{\mathbf{g}}$ with respect to the connection Γ vanishes, so that (2.2.29) implies that Γ^c_{ab} is the Levi-Civita connection for the metric $\widetilde{\mathbf{g}}$.

We now express the field equations (2.2.17) in the new frame described by the metric $\widetilde{\mathbf{g}}$. To this end we write (2.2.17) as $f' R^{am} R_{mb} - \frac{1}{4} f \delta^a_b = 0$ and multiply it by P_{ac} to obtain

$$f' R_{bc} - \frac{1}{4} f P_{bc} = 0. \tag{2.2.30}$$

Since the Ricci tensor depends only on the connection, it remains unchanged in the new frame, *ie* $\widetilde{R}_{ab} = R_{ab}$. Remembering the definition (2.2.27) the previous equation becomes

$$\widetilde{R}_{ab} = \frac{1}{4} \frac{f w}{(f')^2 p^{1/2}} \widetilde{g}_{ab}. \tag{2.2.31}$$

Thus we find again that the Ricci tensor in the conformal frame is proportional to the metric, *ie* we have again an Einstein space.[7]

So far no restriction on f has been imposed, but it turns out that our theory is the simplest one, that is $f(r) = r$. In fact, taking the trace of (2.2.17), we find

$$f'(r)\, r = f(r)\,. \tag{2.2.32}$$

This differential equation is identically satisfied by $f(r) = r$ so that we fall again into the case of a Lagrangian quadratic in the Ricci tensor,

[7] The field equations (2.2.31) having the form $Ric = h\mathbf{g}$, where h is a scalar function, actually represent an Einstein space. In fact, in any manifold $\mathcal{M}$ with $\dim \mathcal{M} \geq 3$, it can be shown using the Bianchi identities that, $Ric = h\mathbf{g}$ implies that h must be a constant.

which was analyzed by Buchdahl. Thus we can set $f'(r) = 1$ in all formulas and rewrite the field equation (2.2.17) in the equivalent form

$$g^{mn}\widetilde{g}_{am}\widetilde{g}_{bn} = \frac{1}{4}g_{ab}g^{rs}g^{mn}\widetilde{g}_{rm}\widetilde{g}_{sn}. \tag{2.2.33}$$

As Buchdahl [23] has pointed out, a solution to this equation is

$$g_{ab} = \psi\widetilde{g}_{ab}, \tag{2.2.34}$$

where ψ is *any* non-zero scalar function. Hence the field equations (2.2.17) and (2.2.20) will be satisfied by any space-time (M, g) conformal to an Einstein space, provided that $p \neq 0$.

Again, given an $f(r)$, (2.2.32) can be regarded as an algebraic equation for the Ricci squared invariant, with roots ρ_1, ρ_2, ... exactly as in the previous paragraph. This gives rise to a whole series of theories, each having a constant value of $R_{ab}R^{ab}$. When this work was completed, we became aware of a preprint by Boroviec *et al* [21] where the $f(r)$ theory is treated in the same spirit as a previous work [43]. The emphasis is again on the algebraic solutions to the trace equation (2.2.32) and it is proved that the field equations reduce to $R_{ab} = \gamma h_{ab}$, where the Ricci tensor is derived from the metric h_{ab}, *ie* the underlined manifold is an Einstein space. However, the constant γ is completely at our disposal. Hence the theory cannot determine uniquely the scalar curvature. (In particular, if one applies an arbitrary $f(r)$ theory in a homogeneous and isotropic universe model in vacuum, one could equally well arrive at a de Sitter or an anti-de Sitter space-time).

For completeness, we show that the introduced metric in (2.2.27) is actually conformally related to the original one. In fact (2.2.17) can be written equivalently as

$$g_{ab} = \frac{1}{4}\frac{w^2 f}{f'^3 p}g^{mn}\widetilde{g}_{am}\widetilde{g}_{bn}. \tag{2.2.35}$$

A solution is

$$g_{ab} = \phi\widetilde{g}_{ab} \text{ with } \phi^2 = \frac{1}{4}\frac{w^2 f}{f'^3 p}. \tag{2.2.36}$$

2.2.3 $f\left(R_{abcd}R^{abcd}\right)$ Lagrangians

As a last example of the uses of the Palatini variation in higher derivative theories, consider the Lagrangian

$$L = f(K), \tag{2.2.37}$$

where $K = R_{abcd}R^{abcd}$. By varying the corresponding action with respect to the metric and the connection, we obtain respectively

$$-\frac{1}{2}fg_{ab} - f^{'}R_a{}^{klm}R_{bklm} + f^{'}R^k{}_{alm}R_{kb}{}^{lm} + 2f^{'}R^k{}_{lam}R_k{}^l{}_b{}^m = 0 \tag{2.2.38}$$

and

$$\nabla_d\left(wf^{'}R_a{}^{(bc)d}\right) = 0. \tag{2.2.39}$$

Again taking the trace of the field equation (2.2.38) we find

$$f^{'}(K)\,K = f(K). \tag{2.2.40}$$

Hence either $f(K) = K$ identically or, given a function f, equation (2.2.40) must be solved algebraically for K.

In contrast to the previous examples there exists no natural way to derive a metric $\widetilde{\mathbf{g}}$ from the field equation (2.2.39) as was the case with the corresponding equations (2.2.6) and (2.2.20) unless the Weyl tensor vanishes [37].

2.2.4 Comments on the Palatini variation

In principle there is no theoretical selection rule to pick out the correct theory between the two derived from the same Lagrangian via the two variational principles. However, the examples discussed in the previous Section bring to light certain constraints on the function f imposed by the Palatini variation, in sharp contrast to the Hilbert variation. Moreover, PV is very restrictive as to the choice of the matter Lagrangian, *ie* for general matter Lagrangians the corresponding field equations are inconsistent in general. Some of the inconsistencies are already encountered at the level of GR and this suggests that the Palatini variation cannot be taken generally as a reliable method.

In the following we recapitulate some of the problems related to the Palatini variation.

- One of the simplest examples is the massless vector field with Lagrangian

$$L = R + \kappa \nabla_a A_b \nabla^a A^b, \tag{2.2.41}$$

where κ is a coupling constant. It can be shown [58] that the field equations obtained from the Lagrangian (2.2.41) by the Palatini method exhibit several kinds of causal anomalies for the propagation of discontinuities of A.

- As we saw in Section 2.2 the Palatini method does not accept general Lagrangians of the form $L = f(r)$ where r is a curvature invariant. It forces the theory to be purely quadratic, ie $f(r) = r$ where r stands for R, $(Ric)^2$ or $(Riem)^2$ (see (2.2.12), (2.2.32) and (2.2.40)). In these cases the Lagrangian density is invariant under conformal transformations $\mathbf{g} \to \phi\mathbf{g}$, where ϕ is an arbitrary function. Since the Γ^c_{ab} and the g_{ab} are unrelated, the corresponding field equations are also conformally invariant. Hence, if $\mathbf{g}$ is any solution to the equations generated by any quadratic Lagrangian, then so is $\phi\mathbf{g}$. This fact is in sharp contrast to the situation encountered in the Hilbert variation. In the curvature squared case, the field equations are identically satisfied if $R = 0$. Hence only one condition is imposed on the 40 functions Γ^c_{ab} while the g_{ab} remain arbitrary. Analogous is the situation in the Ricci squared theory, where the field equations are satisfied by any Ricci flat space leaving the metric quite undetermined. This degree of arbitrariness reflects serious reservations on the use of the Palatini variation in a general arbitrary gravity theory. A common feature of both the R^2 and $(Ric)^2$ field equations is that they are (in the case $R \neq 0$ and $Ric \neq 0$ respectively) conformally equivalent to an Einstein space in which the initially given connection is the Levi-Civita one (see the discussion after (2.2.11) and (2.2.29)). Consequently the field equations are satisfied by any (semi-)Riemannian space conformal to an Einstein space.

- As discussed in Section 2.2, the trace equations (2.2.12) and (2.2.32) can be thought of as algebraic equations which, given a smooth function $f(r)$, give rise to a whole series of Einstein spaces with cosmological constants depending on the roots of the trace equations. However, the 'universality of the Einstein equations' claimed in [43] cannot been taken seriously in general since even in the simplest cases severe inconsistencies emerge. For example, the field equations derived by the Palatini method from the Lagrangian $L = a + bR + cR^2$, $a \neq 0$, lead to a contradiction as Buchdahl has pointed out [23].

- The situation is even worse if matter fields are included in the Lagrangian $L = f(R)$. In that case the stress-energy tensor no longer satisfies the conservation equation. The remedy is to define a new stress-energy tensor which is conserved (see [91] for the curvature squared Lagrangian and a generalization in [49]). However, the physical interpretation of this generalized conservation law is put in doubt. In the weak field limit the equation of motion of test particles derived from the 'conservation equation' exhibits undesirable terms which disagree with Newton's law [49].

2.3 Constrained Palatini Variation

In contrast to Stephenson's and Yang's attempts mentioned above the only consistent way to consider independent variations of the metric and the connection in the context of Riemannian geometry is to add the compatibility condition between the metric and the connection into the Lagrangian as a constraint with Lagrange multipliers [99]. This method is called the *constrained Palatini variation* (CPV). In the case of vacuum general relativity it turns out that the Lagrange multipliers vanish identically as a consequence of the field equations [86]. This result expresses that the Palatini and Hilbert variations are equivalent in vacuum by a mere coincidence. CPV has also been used as a tool to include torsion in the context of general relativity [52]. Safko and Elston [89] applied the CPV to quadratic Lagrangians with the aim of developing a Hamiltonian formulation for these theories.

We consider the most general higher order Lagrangian with general matter couplings

$$L\left(g,\nabla g,...,\nabla^{(m)}g;\,\psi,\nabla\psi,...,\nabla^{(p)}\psi\right). \tag{2.3.1}$$

For an arbitrary symmetric connection Γ^a_{bc} the following identity holds

$$\Gamma^c_{ab}=\left\{{c \atop ab}\right\}-\frac{1}{2}g^{cm}\left(\nabla_b g_{am}+\nabla_a g_{mb}-\nabla_m g_{ab}\right). \tag{2.3.2}$$

For example, in Weyl geometry which is characterized by the relation

$$\nabla_c g_{ab}=-Q_c g_{ab}, \tag{2.3.3}$$

where Q_c is the Weyl covariant vector field, the identity (2.3.2) becomes

$$\Gamma^c_{ab}=\left\{{c \atop ab}\right\}+\frac{1}{2}g^{cm}\left(Q_b g_{am}+Q_a g_{mb}-Q_m g_{ab}\right). \tag{2.3.4}$$

On introducing the difference tensor between the two connections Γ^c_{ab} and $\left\{{c \atop ab}\right\}$

$$C^c_{ab}=\Gamma^c_{ab}-\left\{{c \atop ab}\right\}, \tag{2.3.5}$$

the constrained Palatini variation is defined by adding to the original Lagrangian (2.3.1) the following term as a constraint (with Lagrange multipliers Λ)

$$L_c\left(g,\Gamma,\Lambda\right)=\Lambda_r^{\ mn}\left[\Gamma^r_{mn}-\left\{{r \atop mn}\right\}-C^r_{mn}\right]. \tag{2.3.6}$$

For instance, in Riemannian geometry (2.3.6) takes the form

$$L_c\left(g,\Gamma,\Lambda\right)=\Lambda_r^{\ mn}\left[\Gamma^r_{mn}-\left\{{r \atop mn}\right\}\right],$$

while, in Weyl geometry

$$L_c\left(g,\Gamma,\Lambda\right)=\Lambda_r^{\ mn}\left[\Gamma^r_{mn}-\left\{{r \atop mn}\right\}-\frac{1}{2}g^{rs}\left(Q_n g_{ms}+Q_m g_{sn}-Q_s g_{mn}\right)\right].$$

We then express all the covariant derivatives appearing in (2.3.1) in terms of the connection Γ through the identity (2.3.2), viz.

$$L=L\left(g;\,\Gamma,\nabla\Gamma,...,\nabla^{(m-1)}\Gamma;\,\psi,\nabla\psi,...,\nabla^{(p)}\psi\right)$$

and vary the resulting action

$$S=\int w\left[L\left(g,\Gamma,\psi\right)+L_c\left(g,\Gamma,\Lambda\right)\right] \tag{2.3.7}$$

with respect to the independent fields g, Γ, Λ and ψ. We find

Theorem 2.1. *The field equations obtained from the general Lagrangian (2.3.1) upon Hilbert variation are identical to the field equations derived from the corresponding constrained Lagrangian*

$$L'(g,\Gamma,\Lambda,\psi) = L(g,\Gamma,\psi) + L_c(\Lambda,\Gamma) \tag{2.3.8}$$

upon the constrained Palatini variation.

Proof. (Without loss of generality and to keep the notation as simple as possible we give the proof for Weyl geometry. Generalization to arbitrary symmetric connection is straightforward). Variation with respect to the multipliers ${\Lambda_c}^{ab}$ is straightforward and yields the Weyl condition (2.3.4). Now we proceed to the variation of the action (2.3.7) with respect to the metric g_{ab} taking into account that the constraint (2.3.6) is weakly vanishing. We have first

$$\left.\frac{\delta S}{\delta g^{ab}}\right|_{\Gamma} = \int \left\{ \left.\frac{\delta(wL)}{\delta g^{ab}}\right|_{\Gamma} + \left.\frac{\delta(wL_c)}{\delta g^{ab}}\right|_{\Gamma} \right\}. \tag{2.3.9}$$

Using a local geodesic frame one can easily prove the following relation

$$\begin{aligned} \delta_g \left\{ {}^{c}_{ab} \right\} &= \frac{1}{2}\left(\nabla_b g_{am} + \nabla_a g_{mb} - \nabla_m g_{ab}\right)\delta g^{cm} + \\ &\quad \frac{1}{2} g^{cm}\left[\nabla_b(\delta g_{am}) + \nabla_a(\delta g_{mb}) - \nabla_m(\delta g_{ab})\right] \\ &= -\frac{1}{2}\left(Q_b g_{am} + Q_a g_{mb} - Q_m g_{ab}\right)\delta g^{cm} + \\ &\quad \frac{1}{2} g^{cm}\left[\nabla_b(\delta g_{am}) + \nabla_a(\delta g_{mb}) - \nabla_m(\delta g_{ab})\right]. \end{aligned} \tag{2.3.10}$$

On the other hand, variation of (2.3.5) yields

$$\begin{aligned} \delta_g C^c_{ab} &= \frac{1}{2}\left(Q_b g_{am} + Q_a g_{mb} - Q_m g_{ab}\right)\delta g^{cm} + \\ &\quad \frac{1}{2} g^{cm}\left[Q_b \delta g_{am} + Q_a \delta g_{mb} - Q_m \delta g_{ab}\right]. \end{aligned} \tag{2.3.11}$$

Consequently the variation of the second term on the r.h.s. of (2.3.9) becomes

$$\begin{aligned} \int \delta_g(wL_c) = -\tfrac{1}{2}\int w\,\Lambda^{smn} \; & \left[\nabla_m(\delta g_{sn}) + \nabla_n(\delta g_{ms}) - \nabla_s(\delta g_{mn}) \right. \\ & \left. + Q_m \delta g_{sn} + Q_n \delta g_{ms} - Q_s \delta g_{mn}\right], \end{aligned}$$

which reads, after integrating by parts,

$$\begin{aligned} \int \delta_g(wL_c) \;&= \tfrac{1}{2}\int \; \{\left[\nabla_m(w\,\Lambda^{smn}) - w\,Q_m \Lambda^{smn}\right]\delta g_{sn} + \\ &\quad \left[\nabla_n(w\,\Lambda^{smn}) - w\,Q_n \Lambda^{smn}\right]\delta g_{ms} - \\ &\quad \left[\nabla_s(w\,\Lambda^{smn}) - w\,Q_s \Lambda^{smn}\right]\delta g_{mn}\}. \end{aligned} \tag{2.3.12}$$

Taking into account that $\nabla_a w = -2Q_a w$ the last result becomes

$$\begin{aligned}\int\ \delta_g\left(wL_c\right) \ &= \tfrac{1}{2}\int w \quad \{[\nabla_m\Lambda^{smn} - 3Q_m\Lambda^{smn}]\,\delta g_{sn} + \\ &\qquad\qquad [\nabla_n\Lambda^{smn} - 3Q_n\Lambda^{smn}]\,\delta g_{ms} - \\ &\qquad\qquad [\nabla_s\Lambda^{smn} - 3Q_s\Lambda^{smn}]\,\delta g_{mn}\}\,.\end{aligned} \tag{2.3.13}$$

Rearranging indices, we find

$$\left(\nabla_m\Lambda^{bam}\right)\delta g_{ab} = -\left(\nabla^m\Lambda_{bam} + 3Q^m\Lambda_{bam}\right)\delta g^{ab}. \tag{2.3.14}$$

This last result when inserted into (2.3.13) yields finally

$$\int\ \delta_g\left(wL_c\right) = -\frac{1}{2}\int\ w\nabla^m\left[\Lambda_{bam} + \Lambda_{amb} - \Lambda_{mab}\right]\delta g^{ab}. \tag{2.3.15}$$

Therefore extremization of the action (2.3.7) yields the g–equations

$$\left.\frac{\delta\left(wL\right)}{\delta g^{ab}}\right|_{\Gamma} + wB_{ab} = 0, \tag{2.3.16}$$

where B_{ab} is defined as

$$B^{ab} := -\frac{1}{2}\nabla^m\left[\Lambda_{bam} + \Lambda_{amb} - \Lambda_{mab}\right]. \tag{2.3.17}$$

On the other hand variation with respect to the connection $\Gamma^c{}_{ab}$ yields the Γ–equations

$$\left.\frac{\delta L}{\delta\Gamma^c{}_{ab}}\right|_g + \Lambda_c{}^{ab} = 0. \tag{2.3.18}$$

Finally, variation with respect to the matter fields ψ yields their respective equations of motion. Solving explicitly the Γ–equations (2.3.18) for the multipliers Λ and substituting back the resulting expression into the g–equations, B^{ab} can be written explicitly as

$$B_{ab} = \frac{1}{2}\nabla^m\left[g_{br}g_{ms}\left.\frac{\delta L}{\delta\Gamma^a{}_{rs}}\right|_g + g_{ar}g_{ms}\left.\frac{\delta L}{\delta\Gamma^b{}_{rs}}\right|_g - g_{ar}g_{bs}\left.\frac{\delta L}{\delta\Gamma^m{}_{rs}}\right|_g\right], \tag{2.3.19}$$

while the g–equations (2.3.16) become

$$\left.\frac{\delta\left(wL\right)}{\delta g^{ab}}\right|_{\Gamma} + \frac{1}{2}w\nabla^m\left[g_{br}g_{ms}\left.\frac{\delta L}{\delta\Gamma^a{}_{rs}}\right|_g + g_{ar}g_{ms}\left.\frac{\delta L}{\delta\Gamma^b{}_{rs}}\right|_g - g_{ar}g_{bs}\left.\frac{\delta L}{\delta\Gamma^m{}_{rs}}\right|_g\right] = 0. \tag{2.3.20}$$

We claim that equations (2.3.20) are identical to the field equations obtained from the original Lagrangian (2.3.1) upon the usual Hilbert variation. Indeed, starting from the Hilbert equations in the form

$$\frac{\delta\,(wL)}{\delta g^{ab}} = 0, \tag{2.3.21}$$

and expanding the functional derivative, we obtain

$$\frac{\delta\,(wL)}{\delta g^{ab}} = \left.\frac{\delta\,(wL)}{\delta g^{ab}}\right|_{\Gamma} + \left.\frac{\delta\,(wL)}{\delta \Gamma^{r}{}_{mn}}\right|_{g} \frac{\delta}{\delta g^{ab}}\left[\{{}^{r}_{mn}\} + C^{r}_{mn}\right]. \tag{2.3.22}$$

Using (2.3.10) and (2.3.11) and integrating by parts the action functional corresponding to (2.3.22), we find that the second term on the r.h.s. of (2.3.22) becomes,

$$\begin{aligned}
&\left.\frac{\delta\,(wL)}{\delta \Gamma^{r}{}_{mn}}\right|_{g} \delta_g\left[\{{}^{r}_{mn}\} + C^{r}_{mn}\right] \\
&= \left[-\tfrac{1}{2}\nabla_n\left(wg^{rs}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g}\right)\delta g_{ms} + \nabla_m\left(wg^{rs}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g}\right)\delta g_{sn}\right. \\
&\quad \left. -\nabla_s\left(wg^{rs}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g}\right)\delta g_{mn}\right] + \tfrac{1}{2}w\left(Q_n\delta g_{ms} + Q_m\delta g_{sn} - Q_s\delta g_{mn}\right)g^{rs}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g} \\
&= -\frac{1}{2}\left[\nabla_n\left(wg^{rs}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g}\right)g_{ma}g_{sb} + \nabla_m\left(wg^{rs}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g}\right)g_{sa}g_{nb}\right. \\
&\qquad \left. -\nabla_s\left(wg^{rs}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g}\right)g_{ma}g_{nb}\right]\delta g^{ab} \\
&\quad -\tfrac{1}{2}w\left(Q_n g_{ma}\delta^{r}_{b} + Q_m g_{nb}\delta^{r}_{a} - Q^{r}g_{ma}g_{nb}\right)\delta g^{ab}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g} \\
&= \tfrac{1}{2}w\left[\nabla_n\left(g_{ma}\left.\frac{\delta L}{\delta\Gamma^{b}{}_{mn}}\right|_{g}\right) + \nabla_m\left(g_{nb}\left.\frac{\delta L}{\delta\Gamma^{a}{}_{mn}}\right|_{g}\right) - \nabla^{r}\left(g_{ma}g_{nb}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g}\right)\right]\delta g^{ab} \\
&\quad -\tfrac{1}{2}w\left(Q_n g_{ma}\left.\frac{\delta L}{\delta\Gamma^{b}{}_{mn}}\right|_{g} + Q_m g_{nb}\left.\frac{\delta L}{\delta\Gamma^{a}{}_{mn}}\right|_{g}\right)\delta g^{ab}.
\end{aligned} \tag{2.3.23}$$

On the other hand (2.3.19) is equivalent to

$$\begin{aligned}
B_{ab} &= \frac{1}{2}\left[\nabla_n\left(g_{ma}\left.\frac{\delta L}{\delta\Gamma^{b}{}_{mn}}\right|_{g}\right) + \nabla_m\left(g_{nb}\left.\frac{\delta L}{\delta\Gamma^{a}{}_{mn}}\right|_{g}\right) - \nabla^{r}\left(g_{ma}g_{nb}\left.\frac{\delta L}{\delta\Gamma^{r}{}_{mn}}\right|_{g}\right)\right. \\
&\qquad \left. -Q_n g_{ma}\left.\frac{\delta L}{\delta\Gamma^{b}{}_{mn}}\right|_{g} - Q_m g_{nb}\left.\frac{\delta L}{\delta\Gamma^{a}{}_{mn}}\right|_{g}\right]\delta g^{ab}.
\end{aligned} \tag{2.3.24}$$

A direct comparison of expressions (2.3.24) and (2.3.23) shows the equivalence of the field equations (2.3.20) and (2.3.21). □

The above theorem holds for the Riemannian and Weyl geometries as well for it assumes an arbitrary symmetric connection. In the next Section we shall apply this theorem to the Lagrangian $L = f(R)$ in the framework of Weyl geometry. In the following we illustrate how the CPV works for specific Lagrangians in the framework of Riemannian geometry. We first apply it to quadratic Lagrangians and correct the results obtained in [89]. The g–equations, Γ–equations and the explicit expressions for the B^{ab} tensor are respectively as follows :
for the Lagrangian $L = R^2$,

$$\begin{aligned} & -2RR^{ab} + \frac{1}{2}R^2 g^{ab} + B^{ab} = 0 \\ & \Lambda_c^{\ ab} = \left(2g^{ab}\delta_c^m - g^{am}\delta_c^b - g^{mb}\delta_c^a\right)\nabla_m R \\ & B^{ab} = -2g^{ab}\Box R + 2\nabla^b\nabla^a R, \end{aligned} \tag{2.3.25}$$

for the Lagrangian $L = R_{mn}R^{mn}$,

$$\begin{aligned} & \frac{1}{2}R_{mn}R^{mn}g^{ab} - R^{am}R^b_{\ n} - R_m^{\ b}R^{ma} + B^{ab} = 0 \\ & \Lambda_c^{\ ab} = 2\nabla_c R^{ab} - \delta_c^a\nabla_m R^{mb} - \delta_c^b\nabla_m R^{am} \\ & B^{ab} = -\Box R^{ab} + 2\nabla_m\nabla^b R^{am} - g^{ab}\nabla_n\nabla_m R^{mn} \end{aligned} \tag{2.3.26}$$

and finally, for the Lagrangian $L = R_{mnrs}R^{mnrs}$,

$$\begin{aligned} & \frac{1}{2}R_{mnrs}R^{mnrs}g^{ab} - 2R^{amnr}R^b_{\ mnr} + B^{ab} = 0 \\ & \Lambda_a^{\ bc} = 2\nabla_m R_a^{\ bcm} + 2\nabla_m R_a^{\ cbm} \\ & B^{ab} = 4\nabla_n\nabla_m R^{ambn}. \end{aligned} \tag{2.3.27}$$

Now, if we add the first and the third equations in each of these above cases, we recover the usual field equations inferred by the Hilbert variation. They are respectively (for each quadratic Lagrangian defined above) as follows:

$$\begin{aligned} & \frac{1}{4}R^2 g^{ab} - RR^{ab} + \nabla^b\nabla^a R - g^{ab}\Box R = 0 \\ & \frac{1}{2}R_{mn}R^{mn}g^{ab} - 2R^{bman}R_{mn} + \nabla^b\nabla^a R - \Box R^{ab} - \frac{1}{2}\Box R g^{ab} = 0 \\ & \frac{1}{2}R_{mnrs}R^{mnrs}g^{ab} - 2R^{mnrb}R_{mnr}^{\ \ \ \ a} + 4\nabla_n\nabla_m R^{amnb} = 0. \end{aligned} \tag{2.3.28}$$

As it can be seen by performing the usual metric variation in order to get the fourth order field equations above, CPV has the following advantage:

it simplifies considerably the computations since one deals with a first order formalism rather than a second order one. As a second example we apply CPV to the Lagrangian $L = f(R)$ in the framework of Riemannian geometry. The g–equations, Γ–equations and the explicit expressions for the B^{ab} tensor are respectively as follows :

$$\begin{aligned} & f^{'} R_{(ab)} - \frac{1}{2} f g_{ab} = B_{ab} \\ & \left(2g^{bc}\delta_a^m - g^{mc}\delta_a^b - g^{bm}\delta_a^b\right) \nabla_m f^{'} = \Lambda_a{}^{bc} \\ & B^{ab} = -g^{ab}\Box f^{'} + \nabla^b \nabla^a f^{'} . \end{aligned} \tag{2.3.29}$$

As in the quadratic cases above, if we add the first and the third equations above, we recover the usual field equations obtained by the Hilbert variation, namely

$$f^{'} R_{(ab)} - \frac{1}{2} f g_{ab} - \nabla_a \nabla_b f^{'} + g_{ab} \Box f^{'} = 0.$$

2.4 Conformal structure and Weyl geometry

As discussed in the previous Section, in Weyl geometry the constraint (2.3.6) becomes

$$L_c(\Lambda, C) = \Lambda_c^{ab} \left[\Gamma^c_{ab} - \left\{ {c \atop ab} \right\} - \frac{1}{2} g^{cm} \left(Q_a g_{mb} + Q_b g_{am} - Q_m g_{ab} \right) \right] . \tag{2.4.1}$$

In order to examine the consequences of the Weyl constraint (2.4.1), we apply CPV to the Lagrangian $L = f(R)$. Variation with respect to the Lagrange multipliers recovers the expression (2.3.4) of the Weyl connection. Variation with respect to the metric yields the $g-$equations

$$f' R_{(ab)} - \frac{1}{2} f g_{ab} + B_{ab} = 0, \tag{2.4.2}$$

where B_{ab} is defined by (2.3.17). Variation with respect to the connection yields the explicit form of the Lagrange multipliers

$$\Lambda_c^{ab} = \frac{1}{2} \delta_c^b \left(Q^a f' - \nabla^a f' \right) + \frac{1}{2} \delta_c^a \left(Q^b f' - \nabla^b f' \right) - g^{ab} \left(Q_c f' - \nabla_c f' \right) . \tag{2.4.3}$$

Following the prescriptions of the CPV we now substitute back this last result into the definition (2.3.17) of the B_{ab} tensor to obtain

$$\begin{aligned} B_{ab} = \; & 2Q_{(a}\nabla_{b)}f' - \nabla_{(a}\nabla_{b)}f' + f'\nabla_{(a}Q_{b)} - f'Q_aQ_b \\ & -g_{ab}\left(2Q_m\nabla^m f' - Q^2 f' - \Box f' + f'\nabla^m Q_m\right). \end{aligned} \tag{2.4.4}$$

Inserting this result into the $g-$ equations (2.4.2) we find the full field equations for the Lagrangian $L = f(R)$ in the framework of Weyl geometry, namely

$$f'R_{(ab)} - \frac{1}{2}fg_{ab} - \nabla_a\nabla_b f' + g_{ab}\Box f' = M_{ab}, \tag{2.4.5}$$

where M_{ab} is defined by

$$M_{ab} = -2Q_{(a}\nabla_{b)}f' - f'\nabla_{(a}Q_{b)} + f'Q_aQ_b + g_{ab}\left(2Q_m\nabla^m f' - Q^2 f' + f'\nabla^m Q_m\right). \tag{2.4.6}$$

It is interesting to note that the degenerate case $Q_a = 0$ corresponds to the usual field equations obtained by the Hilbert variation in the framework of Riemannian geometry, namely

$$f'R_{ab} - \frac{1}{2}fg_{ab} - \nabla_a\nabla_b f' + g_{ab}\Box f' = 0.$$

It is known that these equations are conformally equivalent to Einstein equations with a self-interacting scalar field as the matter source [6]. In what follows we generalize this property of the $f(R)$ field equations in Weyl geometry. To this end we define the metric $\widetilde{g}$ conformally related to g with f' as the conformal factor. Taking into account that the Weyl vector transforms as

$$\widetilde{Q}_a = Q_a - \nabla_a \ln f',$$

the field equations (2.4.5) in the conformal frame are

$$f'\widetilde{R}_{(ab)} - \frac{1}{2}\frac{f}{f'}\widetilde{g}_{ab} - \widetilde{\nabla}_a\widetilde{\nabla}_b f' + \widetilde{g}_{ab}\widetilde{\Box} f' = \widetilde{M}_{ab},$$

where $\widetilde{\nabla} = \nabla$, $\widetilde{\Box} = \widetilde{g}^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b = (f')^{-1}\Box$ and $\widetilde{M}_{ab}$ is given by

$$\widetilde{M}_{ab} = -f'\widetilde{\nabla}_{(a}\widetilde{Q}_{b)} + f'\widetilde{Q}_a\widetilde{Q}_b - \widetilde{\nabla}_a\widetilde{\nabla}_b f' + \widetilde{g}_{ab}\left(-f'\widetilde{Q}^2 + f'\widetilde{\nabla}^m\widetilde{Q}_m + \widetilde{\Box} f'\right).$$

Defining a scalar field φ by $\varphi = \ln f'$ and the potential in the 'usual' form

$$V(\varphi) = \frac{1}{2}\left(f'(R)\right)^{-2}\left(Rf'(R) - f(R)\right), \tag{2.4.7}$$

we find that the field equations take the final form,

$$\widetilde{G}_{ab} = \widetilde{M}^{Q}_{ab} - \widetilde{g}_{ab} V(\varphi), \tag{2.4.8}$$

where we have defined

$$\widetilde{G}_{ab} := \widetilde{R}_{(ab)} - \frac{1}{2}\widetilde{R}\widetilde{g}_{ab}$$

and

$$\widetilde{M}^{Q}_{ab} := -\widetilde{\nabla}_{(a}\widetilde{Q}_{b)} + \widetilde{Q}_a\widetilde{Q}_b + \widetilde{g}_{ab}\left(-\widetilde{Q}^2 + \widetilde{\nabla}^m\widetilde{Q}_m\right).$$

The field equations (2.4.8) could also be obtained from the corresponding conformally transformed Lagrangian using the CPV. In that case the equation of motion of the scalar field φ derived upon variation with respect to φ would imply that the potential $V(\varphi)$ is constant.

Formally the field equations (2.4.8) look like Einstein equations with a cosmological constant and a source term $\widetilde{M}^{Q}_{ab}$ depending on the field $\widetilde{Q}_a$. However, they cannot be identified as such unless the geometry is Riemannian, *ie* $\widetilde{Q}_a = 0$. This will be the case only if the original Weyl vector is a gradient, $Q_a = \nabla_a \Phi$. Then it can be gauged away by the conformal transformation $\widetilde{g}_{ab} = (\exp\Phi)\, g_{ab}$ and therefore the original space is not a general Weyl space but a Riemann space with an undetermined gauge [93]. It is exactly what happens when the PV is applied to the Lagrangian $L = f(R)$. In that case the Weyl vector can be deduced using (2.2.7) and turns out to be $Q_a = \nabla_a(\ln f')$. This fact shows that the PV is not the most general metric-affine variational method for it cannot deal with a general Weyl geometry. Moreover it can be considered as a degenerate case of the CPV in Weyl geometry, for the field equations obtained from the former can be recovered only by choosing a specific Weyl vector.

2.5 Summary

The theorem stated in Section 2.3 has the interpretation that the only consistent way to investigate generalized theories of gravity without imposing from the begining that the geometry is Riemannian, is the constrained Palatini variation. Applications to quadratic Lagrangians and

$f(R)$ Lagrangians in the framework of Riemannian and Weyl geometry, respectively, reveal that (a): PV is a degenerate case corresponding to a particular gauge and (b): The usual conformal structure can be recovered in the limit of vanishing Weyl vector. The generalization of Theorem 2.1 to include arbitrary connections with torsion should be the subject of future research. The physical interpretation of the source term in equation closely related to the choice of the Weyl vector field Q. However, M_{ab} cannot be interpreted as a genuine stress-energy tensor in general since, for instance, choosing Q to be a unit timelike vector field hypersurface orthogonal, the sign of $M_{ab}Q^aQ^b$ depends on the signs of $f'(R)$ and the 'expansion' $\nabla_a Q^a$.

The non-trivial generalization of the conformal equivalence theorem presented in Section 3.3.2 opens the way to analyzing cosmology in the framework of these Weyl $f(r)$ theories by methods such as those used in the traditional Riemannian case. The first steps in such a program may be as follows.

(A) Analyze the structure and properties of Friedman cosmologies, find their singularity structure and examine the possibility of inflation.

(B) Consider the past and future asymptotic states of Bianchi cosmologies. Examine isotropization and recollapse conjectures in such universes. Look for chaotic behavior in the Bianchi VIII and IX spacetimes.

(C) Formulate and prove singularity theorems in this framework. This will not be as straightforward as in the Riemannian case (cf. [6]) because of the presence of the source term M_{ab}.

All the problems discussed above can be tackled by leaving the conformal Weyl vector field $\widetilde{Q}_a$, undetermined while, at the end, setting it to zero will lead to detailed comparisons with the results already known in the Riemannian case.

Part II

Cosmological dynamics

Chapter 3

Cosmic no-hair theorems

In this chapter we present the cosmic no-hair conjecture and give a proof of this for a curvature squared gravity theory for all Bianchi cosmologies. In section 3.1, after a brief discussion of the idea of inflation, we review the main properties of de Sitter spacetime which are needed for the subsequent development. In section 3.2 we carefully present the cosmic no-hair conjecture and discuss its limitations. We review the CNHC in several inflationary models, thus setting the outline of its validity. In the last section we give the proof of the main result of this chapter, namely that, in the conformal frame of the $R + \beta R^2 + L_{matter}$ theory, *all* expanding Bianchi universes asymptotically approach de Sitter spacetime provided certain reasonable conditions on matter hold.

3.1 The inflationary scenario and de Sitter spacetime

As discussed in Chapter One, standard cosmology has a difficulty in explaining the observed isotropy of the universe – unless one *assumes* that isotropy persists back to the big bang. Besides the horizon problem discussed in the introduction, there are some other interrelated problems of the standard cosmology, like the monopole problem, the flatness problem and the problem of the small-scale inhomogeneity of the universe (see [67] for a discussion).

These problems led to the invention in 1981 of the inflationary scenario which is a modification of the standard hot big bang model. According to this scenario the very early universe underwent a short period of exponential expansion, or inflation, during which its radius increased by a factor of about 10^{50} times greater than in standard cosmology. This inflationary phase is also known as the de Sitter phase since de Sitter universe is an homogeneous and isotropic universe with radius growing exponentially with time. From times later than about 10^{-30} sec the history of the universe is described by the standard cosmology and all the successes of the later are maintained.

To see what this picture implies for our universe, consider a region which at time t as measured from the big bang has the size of the horizon distance at time t. The horizon distance at time t is approximately the distance travelled by light in time t, *ie* equals ct. This is evidently the greatest size of a causally coherent region possible. Thus at time $10^{-34} sec$ when inflation commences the size of this region is about $10^{-24} cm$. After inflation, at time about $10^{-30} sec$, its size has grown to approximately $10^{26} cm$. The observable universe at that epoch had a radius of about $10 cm$, a minuscule part of the inflating region. Since the universe lay within a region which started as a causally coherent one, it would have had time to homogenize and isotropize. Thus inflation naturally explains the uniformity of the universe.

Today inflation remains the only way to approach a solution of most problems of standard cosmology. It must be emphasized that the inflationary scenario is far from being a complete theory describing the very early universe. Several inflationary models have been developed during the last fifteen years, mainly because there exist different ways to generate the mechanism of inflation. These models have problems of their own. Inflation remains an area of active research [68].

In all inflationary models inflation is driven by the false vacuum energy of a scalar field. The field has a potential with a flat portion and slowly rolls down the potential curve. One usually assumes that the scalar field, φ, is minimally coupled to gravity with a Lagrangian

$$L = -\frac{1}{2}\nabla_a\varphi\nabla^a\varphi - V\left(\varphi\right).$$

Therefore the stress-energy tensor of the scalar field is

$$T_{ab}^{\varphi} = \nabla_a \varphi \nabla_b \varphi - \frac{1}{2} g_{ab} \left[\nabla_c \varphi \nabla^c \varphi + 2V(\varphi) \right]. \tag{3.1.1}$$

We now can see the effect of a scalar field in driving the universe to exponential expansion. Assume for simplicity that the scalar field is constant. This is a good approximation as far as the field belongs in the interval where the potential is flat. A constant scalar field φ over all spacetime simply represents a 'restructuring of the vacuum' in the sense that the vacuum energy density changes by a quantity proportional to $V(\varphi)$. If there were no gravitational effects this change would be unobservable, but in general relativity it affects the properties of spacetime. In fact $V(\varphi)$ enters into the Einstein equation in the following way:

$$G_{ab} = T_{ab} \equiv T_{ab}^{m} - g_{ab} V, \tag{3.1.2}$$

where T_{ab} is the total stress-energy tensor, T_{ab}^{m} is the stress-energy tensor of ordinary matter and $-g_{ab}V$ is the stress-energy tensor of the vacuum (the constant scalar field). Of course $V(\varphi) = V$ is a constant. Note that (3.1.2) is just the Einstein equation with a cosmological constant Λ, viz.

$$G_{ab} = T_{ab}^{m} - \Lambda g_{ab}, \tag{3.1.3}$$

where Λ is given by

$$\Lambda = V(\varphi). \tag{3.1.4}$$

We may also view (3.1.2) or (3.1.3) in vacuum ($T_{ab}^{m} = 0$) as describing a perfect fluid with pressure $q := -E = -V$. This large negative pressure has the effect that a homogeneous and isotropic universe expands exponentially. Since in most inflationary models E is of order $\sim 10^{73} g/cm^3$, we see that a FRW universe expands almost exponentially with a minuscule time constant of order $H^{-1} \sim 10^{-34}$ sec [17]. The time[1] required for φ to evolve to its equilibrium value is $\sim 100 H^{-1}$. Hence the universe has enough time to inflate.

We now briefly review the basic properties of de Sitter spacetime since it plays a central role in inflation (for a more complete discussion see HE [50] and [92]). de Sitter spacetime can be defined as the maximally symmetric vacuum solution of the Einstein equations. Recall that

[1] This time of course is model dependent.

the spacetimes of constant curvature are locally characterized by the condition

$$R_{abcd} = \frac{1}{12} R \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) \tag{3.1.5}$$

which implies that

$$R_{ab} = \frac{1}{4} R g_{ab}. \tag{3.1.6}$$

The second Bianchi identities, $R_{ab[cd;e]} = 0$, imply that R is covariantly constant, *ie*

$$R = R_0 = const. \tag{3.1.7}$$

The Einstein tensor is therefore $R_{ab} - \frac{1}{2} R g_{ab} = -\frac{1}{4} R_0 g_{ab}$. Hence one can regard these spaces as solutions of the vacuum Einstein equations with $\Lambda = \frac{1}{4} R_0$. Alternatively these spaces can be thought of as solutions of the Einstein equations with a perfect fluid having constants density and pressure $-p = \rho = \Lambda$. (If $R_0 = 0$, the solution is Minkowski spacetime. The de Sitter spacetime is the only maximally symmetric *curved* spacetime).

de Sitter space is more easily visualized as the pseudosphere $S_1^4 \left(H^{-1} \right)$ [82]

$$-\left(x^0 \right)^2 + \left(x^1 \right)^2 + \left(x^2 \right)^2 + \left(x^3 \right)^2 + \left(x^4 \right)^2 = H^{-2} \tag{3.1.8}$$

embedded in a five-dimensional Lorentz space. The radius of the pseudosphere $S_1^4 \left(H^{-1} \right)$ is $H^{-1} = \sqrt{3/\Lambda}$. In coordinates (t, χ, θ, ϕ) defined on the pseudosphere by

$$\begin{aligned} x^0 &= H^{-1} \sinh Ht \\ x^1 &= H^{-1} \cosh Ht \cos \chi \\ x^2 &= H^{-1} \cosh Ht \sin \chi \cos \theta \\ x^3 &= H^{-1} \cosh Ht \sin \chi \cos \theta \\ x^4 &= H^{-1} \cosh Ht \sin \chi \sin \theta \sin \phi \end{aligned} \tag{3.1.9}$$

the de Sitter metric takes the usual form of a FRW universe

$$ds^2 = -dt^2 + H^{-2} \cosh^2 (Ht) \left[d\chi^2 + \sin^2 \chi \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \tag{3.1.10}$$

with topology $\mathbb{R} \times \mathbb{S}^{\not\Vdash}$.

With the introduction of coordinates (t, x, y, z) on the pseudosphere by

$$\begin{aligned} t &= H^{-1} \ln H \left(x^0 + x^1\right) \\ x &= H^{-1} x^2 / \left(x^0 + x^1\right) \\ y &= H^{-1} x^3 / \left(x^0 + x^1\right) \\ z &= H^{-1} x^4 / \left(x^0 + x^1\right), \end{aligned} \tag{3.1.11}$$

the metric takes the form

$$ds^2 = -dt^2 + \exp\left(2Ht\right)\left(dx^2 + dy^2 + dz^2\right). \tag{3.1.12}$$

Other framings of de Sitter spacetime can be found in [92].

3.2 Cosmic no-hair conjecture

In principle the inflationary scenario provides an explanation of the homogeneity and isotropy of the universe without assuming this symmetry as part of the initial conditions. However, many constructions of inflationary models incorporate homogeneity and isotropy from the outset. In the previous section we analyzed inflation in the context of an FRW cosmology assuming that the inflating regions are smooth enough so that they can be regarded as de Sitter spacetimes.

It is not obvious that cosmological models with non-FRW initial conditions ever enter an inflationary epoch or that, if inflation occurs, initial inhomogeneities and anisotropies will be smoothed out eventually. Therefore the question of the naturalness of the inflationary scenario is posed in the sense that we have to ask: Did the inflationary phase in the evolution of the universe proceed from very general initial conditions?

With regard to the question of whether the universe evolves to a homogeneous and isotropic state during an inflationary epoch Gibbons and Hawking [45] and Hawking and Moss [51] have put forward the following

Conjecture 3.1. *All expanding-universe models with positive cosmological constant asymptotically approach the de Sitter solution.*

This is referred to as *the cosmic no-hair conjecture.*

In general relativity solutions of the Einstein equations are believed to settle toward stationarity as the nonstationary parts dissipate in the form of gravitational radiation [77]. Such a proposition is very difficult to prove, even for the simplest spacetimes. Even if we accept this principle, neither the final state of evolution nor its uniqueness is at all obvious. Such uniqueness assertions are known as 'no-hair conjectures', to denote the loss of information regarding initial spacetime geometry, caused by evolution under the field equations. This information either radiates out to infinity or is hidden behind event horizons. The cosmic no-hair conjecture is an assertion of the uniqueness of the de Sitter metric as a stationary,[2] no-black-hole solution of the Einstein equation with positive cosmological constant.

A few comments about the cosmic no-hair conjecture (CNHC) are necessary.

- A precise version of this conjecture is difficult to formulate, mainly because of the vagueness associated with the terms 'asymptotic approach' and 'expanding universe' [102].

- There is no general proof (or disproof) of this conjecture. In the case of an arbitrary spacetime where the full Einstein equations hold the validity of this conjecture is completely unknown (see, however, [77]).

- Some counter-examples exist of the form 'initially expanding universe models recollapse to a singularity' without ever becoming de Sitter type universes, the most obvious one being the closed FRW spacetime which collapses before it enters an inflationary phase (see [3, 8]).

- It should be possible for regions of the universe to collapse to black holes so that the universe approaches a de Sitter solution with black holes rather than a de Sitter solution. In addition other special behaviors should be possible such as an asymptotic approach to an Einstein static universe [102].

- Although the CNHC is not generally valid as it stands, the number and diversity of the models that do obey this principle lead to the belief that perhaps a weaker version of the conjecture must be true.

[2] The stationary nature of de Sitter spacetime is discussed in [77, 92].

Some evidence for the CNHC has been discussed by Boucher and Gibbons [20], and Barrow [2]. They studied small perturbations of de Sitter spacetime and found that they do not grow as the scale factor tends to infinity. Steigman and Turner [97] considered a perturbed FRW model dominated by shear or negative curvature when inflation begins in the context of new inflation. They found that the size of a causally coherent region after inflation is only slightly smaller than the usual one in a purely FRW model. Wald [102] was the first who succeeded in 1983 to prove that *'all expanding Bianchi cosmologies with positive cosmological constant* Λ*, except type-IX, evolve towards the de Sitter solution exponentially fast. The behavior of type-IX models is similar provided that* Λ *is greater than a certain bound'*. Wald's proof is the prototype for many subsequent works [57, 84]. For possible generalizations to inhomogeneous spacetimes see [77, 80].

We can show by a physical argument why Wald's theorem is to be expected [60]. For this purpose we need only the time-time component of the Einstein equation as our analysis will be quite qualitative. Denoting the scale factors of the three principal axes of the universe by X_i, $i = 1, 2, 3$ and the mean scale factor by $a = V^{1/3}$, where $V = X_1 X_2 X_3$, the time-time component of $G_{ab} = \Lambda g_{ab} + T_{ab}$ is written as

$$3H^2 \equiv 3\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(\frac{\dot{V}}{V}\right)^2 = \Lambda + T_{ab}n^a n^b + F\left(X_1, X_2, X_3\right). \qquad (3.2.1)$$

The function F depends on the Bianchi model and contains all the information about the anisotropic expansion of the mean scale factor. The detailed form of F can be found in [41]. The only property of F that we will use is that in all Bianchi models F decreases at least as fast as a^{-2}. Equation (3.2.1) is the analog of the Friedmann equation for anisotropic cosmologies. For the FRW models, $X_1 = X_2 = X_3$ and $F = k/a^2$. The term $\dot{V}/V$ in (3.2.1) is the expansion K in Wald's theorem.

As the universe expands, the function F in (3.2.1) decreases at least as a^{-2} and the term $T_{ab}n^a n^b$ decreases as some power of a (for example as a^{-4} for a radiation dominated FRW universe). It is clear that the cosmological constant eventually dominates the terms $T_{ab}n^a n^b$ and F. This happens in about one Hubble time, $H_0^{-1} = (\Lambda/3)^{-1/2}$, and $a \sim \exp H_0 t$.

We see that the anisotropic term F and the ordinary matter content of the universe decay exponentially with time and the spacetime rapidly approaches de Sitter. A careful analysis of the asymptotic approach to de Sitter spacetime can be found in [9].

Wald's theorem is valid in the context of general relativity, assuming a positive cosmological constant. However, in inflationary models the universe does not have a true cosmological constant. Rather there is a vacuum energy density, which during the slow evolution of the scalar field remains approximately constant and behaves like a cosmological term. Therefore we are faced with the question: Does the universe evolve towards a de Sitter type state before the potential energy of the scalar field reaches its minimum? To answer to this question, one has to take into account the dynamics of the scalar field in the no-hair hypothesis.

In many inflationary models the particular form of the potential $V(\varphi)$ of the inflaton field is predicted by some particle theory. An alternative approach is to consider very simple forms of $V(\varphi)$ such as $m^2\varphi^2$ or $\lambda\varphi^4$, not directly related to any particular physical theory [67]. This approach is reasonable since we do not really know which theory of particle physics best describes the very early universe. The CNHC has been examined for homogeneous cosmologies for specific inflationary models, namely new inflation, chaotic inflation, power-law inflation, inflation in the context of HOG theories. The terminology derives mainly from the form of the potential function of the scalar field which drives the inflation.

In chaotic inflation Moss and Sahni [79], using a quadratic potential, proved the NHT for Bianchi-type models except type IX with matter satisfying the SEC and DEC. Heusler [53] generalized this result in the case of any convex potential having a local minimum $V_0 > 0$ (see next Chapter).

Using an exponential potential Halliwell [48] showed that the power-law inflation solution, $a(t) \sim t^p,\ p > 1$, is an attractor for homogeneous and isotropic cosmologies. Ibanez *et al* [55] studied the CNHC for the Bianchi-type I, V, VII_0 or VII_h models without matter, in the spirit of Heusler's theorem. A much more complete discussion of the CNHC for homogeneous cosmologies including type IX in the case of an exponential potential leading to power-law inflation can be found in Kitada and

Maeda [59].

We now turn to the discussion of the CNHC in the context of higher order gravity theories. An interesting feature of HOG theories is that inflation emerges in these theories in a most direct way. In one of the first inflationary models, proposed in 1980 by Starobinsky [95], inflation is due to the R^2 correction term in a gravitational Lagrangian $L = R + \beta R^2$ where β is a constant. Instead of having to rely on the existence of a scalar field, inflation in the present context is driven by the higher order curvature terms present in the Lagrangian without assuming a scalar field at all. Here the role of the scalar field is played by the scalar curvature of the spacetime. The situation is not surprising under the light of the conformal transformation theorem stated in Chapter One. For a review of different inflation theories conformally related, see Gottlöber *et al* [46].

The existence of the de Sitter solution and its stability in $f(R)$ theories has been examined by Barrow and Ottewill [12]. Cotsakis and Flessas [32] have shown that the quasi-de Sitter solution $a(t) \sim \exp(Bt - At^2)$ with A, B constants is an attractor for all homogeneous and isotropic spacetimes in any $f(R)$ theory. Cotsakis and Saich [36] studied the stability of the power-law solution for homogeneous and isotropic models in the context of a $L = R^n$ theory. A cosmic NHT for vacuum homogeneous cosmologies except type IX models in a quadratic theory has been demonstrated by Maeda [71] (see also Mijic & Stein-Schabes [76] and Berkin [16]). The proof is based on the conformal equivalence theorem and thus relies on general relativity dynamics. For a review of the more important works on the CNHC see [74].

3.3 Proof of the CNHT in a curvature squared theory for all Bianchi-type cosmologies

In this section we prove the central result, namely the no-hair theorem in a curvature-squared theory for all Bianchi-type cosmologies (including Bianchi-type IX) with matter content satisfying certain energy conditions.

Consider a four-dimensional spacetime $(M, \mathbf{g})$ and a theory derived from a Lagrangian $L = R + \beta R^2 + L_m$ where L_m denotes the matter Lagrangian. The field equations (1.0.10) take the form

$$R_{ab} - \frac{1}{2} g_{ab} R - \frac{\beta}{1+2\beta R}\left(2\nabla_a \nabla_b R - 2 g_{ab} \Box R + \frac{1}{2} g_{ab} R^2\right) = T_{ab}\left(\mathbf{g}\right). \tag{3.3.1}$$

Under a conformal transformation of the metric (compare to the general formalism of Chapter One)

$$\widetilde{g}_{ab} = \left(1 + 2\beta R\right) g_{ab}, \tag{3.3.2}$$

with

$$\varphi = \sqrt{\frac{3}{2}} \ln\left(1 + 2\beta R\right), \tag{3.3.3}$$

the field equations (3.3.1) become the Einstein equations in the new spacetime $(M, \widetilde{\mathbf{g}})$

$$\widetilde{R}_{ab} - \frac{1}{2} \widetilde{g}_{ab}\, \widetilde{R} = \nabla_a \varphi \nabla_b \varphi - \frac{1}{2} \widetilde{g}_{ab} \left(\nabla_c \varphi \nabla^c \varphi\right) - \widetilde{g}_{ab} V + \widetilde{T}_{ab}\left(\widetilde{\mathbf{g}}\right), \tag{3.3.4}$$

$$\widetilde{\Box}\varphi - V'\left(\varphi\right) = 0, \tag{3.3.5}$$

where the potential V is given by

$$V = \frac{1}{8\beta}\left[1 - \exp\left(-\sqrt{2/3}\varphi\right)\right]^2. \tag{3.3.6}$$

As Maeda [71] has pointed out, this potential has a long and flat plateau. When φ is far from the minimum of the potential, V is almost constant $V_\infty := \lim_{\varphi \to +\infty} V\left(\varphi\right) = 1/\left(8\beta\right)$. Thus V has the general properties for inflation to commence and V_∞ behaves as a cosmological term. (The constant β is of order $10^{14}\ l_{PL}^2$ [73].)

We shall work exclusively in the conformal picture and so for simplicity we drop the tilde. Our starting point is a (spatially) homogeneous spacetime which, according to the results of Appendix B, can be foliated by a one-parameter family of spacelike hypersurfaces Σ_t orthogonal to a congruence of timelike geodesics parametrized with proper time t. As usual we denote by $\mathbf{n} = \partial/\partial t$ the unit tangent vector field to the geodesics. The spatial metric is related to the spacetime metric as $h_{ab} = g_{ab} + n_a n_b$. As discussed in the Appendix B, for homogeneous spacetimes the scalar curvature R becomes a function only of time. Hence the scalar field introduced by (3.3.3) is homogeneous. Moreover, the Einstein equations (3.3.4) become ordinary differential equations with respect to time. Ordinary matter is assumed to satisfy the strong and dominant energy conditions, namely

$$\begin{aligned} T_s &:= \left(T_{ab} - \tfrac{1}{2} T g_{ab}\right) n^a n^b \geq 0 \\ T_d &:= T_{ab} n^a n^b \geq 0 \quad \text{and } T^a{}_b n^b \text{ is non-spacelike} \end{aligned} \tag{3.3.7}$$

for any unit timelike vector field n^a. When it happens that matter is moving along the geodesics defined by $\mathbf{n}$, the expansion, shear and the rotation of the cosmic fluid coincide with the corresponding quantities of the geodesic congruence. However, we do not suppose that this be the case since we can formally treat the scalar field as a perfect fluid with velocity vector field

$$u^a = \frac{\nabla^a \varphi}{\sqrt{-\nabla_a \varphi \nabla^a \varphi}}. \tag{3.3.8}$$

Furthermore, we construct the spacelike homogeneous hypersurfaces in such a way that u^a is hypersurface orthogonal. In other words we identify the vector field $\mathbf{n}$ with the velocity vector field $\mathbf{u}$ of the fluid representing the scalar field φ. With the above choice of $\mathbf{n}$ we have $n^a \nabla_a = \partial / \partial t$. Hence

$$T^{\varphi}_{ab} n^a n^b = E \quad \text{and} \quad \left(T^{\varphi}_{ab} - \frac{1}{2} g_{ab} T^{\varphi}\right) n^a n^b = E + 3q, \tag{3.3.9}$$

where the energy density E and the pressure q of the scalar field are defined by

$$E := \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \quad \text{and} \quad q := \frac{1}{2} \dot{\varphi}^2 - V(\varphi). \tag{3.3.10}$$

We use the Einstein equations (3.3.4) written as

$$G_{ab} = T^{\varphi}_{ab} + T_{ab} \tag{3.3.11}$$

to describe the evolution of Bianchi cosmologies. In what follows besides the equation of motion of the scalar field only two components of (3.3.11) are necessary: the time-time component

$$G_{ab} n^a n^b - E - T_d = 0 \tag{3.3.12}$$

and the 'Raychaudhuri' equation

$$R_{ab} n^a n^b + E + 3q - T_s = 0. \tag{3.3.13}$$

The term $R_{ab}n^an^b$ permits us to transform (3.3.13) to its more familiar form as follows. Firstly we decompose K_{ab} into its trace K and traceless part σ_{ab} (see (A.0.5) and (A.1.4)), viz.

$$K_{ab} = \frac{1}{3}Kh_{ab} + \sigma_{ab}. \tag{3.3.14}$$

Note that by (A.1.3) K is related to the determinant h of the spatial metric as

$$K = \frac{d}{dt}\left(\ln h^{1/2}\right). \tag{3.3.15}$$

We can now express $G_{ab}n^an^b$ in terms of the three-geometry of the homogeneous hypersurface using the Gauss-Codacci equation (see HE, [50])

$$\frac{1}{2}\,{}^{(3)}R = \frac{1}{2}R + R_{ab}n^an^b - \frac{1}{2}\left(K^a{}_a\right)^2 + \frac{1}{2}K_{ab}K^{ab}. \tag{3.3.16}$$

Observe that the sum of the first two terms on the right-hand side of the Gauss-Codacci equation equals $G_{ab}n^an^b$, while the last term of this equation simplifies as $K_{ab}K^{ab} = \frac{1}{3}K^2 + \sigma_{ab}\sigma^{ab}$. Putting all these together in (3.3.12) we obtain

$$\frac{1}{3}K^2 = \sigma^2 + T_d + \frac{1}{2}\,\dot{\varphi}^2 + V - \frac{1}{2}\,{}^{(3)}R \tag{3.3.17}$$

where we set $2\sigma^2 := \sigma_{ab}\sigma^{ab}$.

Eliminating $R_{ab}n^an^b$ from the Raychaudhuri equation (A.0.6) and (3.3.13) we obtain the Raychaudhuri equation with a scalar field

$$\dot{K} = -\frac{1}{3}K^2 - 2\sigma^2 - T_s - \dot{\varphi}^2 + V. \tag{3.3.18}$$

Finally, the equation of motion for the scalar field (3.3.5) becomes

$$\ddot{\varphi} + K\,\dot{\varphi} + V'(\varphi) = 0. \tag{3.3.19}$$

Using the above two components of the Einstein equations and the equation of motion of the scalar field we prove below the following three propositions. Firstly we show that *all Bianchi models which are initially on the flat plateau of the potential (*3.3.6*), except probably Bianchi IX, with a matter content satisfying the strong and dominant energy conditions, rapidly approach de Sitter space-time.* Secondly we show that *Bianchi-type IX also isotropizes if initially the scalar three-curvature* ${}^{(3)}R$

is less than the potential V of the scalar field. Thirdly we show that *the time needed for the potential energy to reach its minimum is much larger than the time of isotropization.* Hence we conclude that the universe reaches the potential minimum (at $\varphi = 0$) at which the cosmological term vanishes and consequently evolves according to the standard Fiedmann cosmology.

In Appendix B it is shown that ${}^{(3)}R \leq 0$ in all Bianchi models except type-IX. We observe that, assuming a negative spatial scalar curvature, all terms in the right-hand side of (3.3.17) are positive and so we turn our attention to any Bianchi model which is not type-IX.

Assume that the initial value φ_i of the scalar field is large and positive, *ie* the universe is on the flat plateau of the potential and that the kinetic energy of the field is negligible when compared to the potential energy. If the universe is initially expanding, *ie* $K > 0$ at some arbitrary time t_i, then (3.3.17) implies that it will expand for all subsequent times, *ie* $K > 0$ for $t \geq t_i$ for all Bianchi models, except possibly type-IX. Multiplying (3.3.19) by $\dot{\varphi}$ we find for the energy density of the scalar field $E = \frac{1}{2}\dot{\varphi}^2 + V$ that

$$\dot{E} = -K\dot{\varphi}^2 . \tag{3.3.20}$$

Hence, in an expanding universe the field looses energy and slowly rolls down the potential. The 'effective' regime of inflation is the phase during the time interval $t_f - t_i$ needed for the scalar field to evolve from its initial value φ_i to a smaller value φ_f, where φ_f is determined by the condition that $V\left(\varphi_f\right) \simeq \eta V_\infty$. The numerical factor η is of order say 0.9, but its precise value is irrelevant. As we shall show, the universe becomes de Sitter space during the effective regime followed by the usual FRW model when the cosmological term vanishes. To this end we define a function S, which plays the same role as in Moss and Sahni [79], by

$$S = \frac{1}{3}K^2 - E. \tag{3.3.21}$$

In all Bianchi models except IX this function is non-negative due to the dominant energy condition and the fact that in these models the scalar spatial curvature is non-positive. Taking the time derivative of S and using eqs. (3.3.17) and (3.3.18) we obtain

$$\dot{S}= -\frac{2}{3}KS - \frac{2}{3}\left(2\sigma^2 + T_s\right)K. \tag{3.3.22}$$

It follows that $\dot{S}\leq -\frac{2}{3}KS$ or

$$\dot{S}\leq -\frac{2}{3}S\sqrt{3\left(S+E\right)}. \tag{3.3.23}$$

This differential inequality cannot be integrated immediately because E is a function of time (albeit slowly-varying). However, as mentioned above, in order to have inflation, E must be bounded from below, *ie* the scalar field must be large enough so that $E \geq \eta V_\infty$. In that case inequality (3.3.23) implies that

$$S \leq \frac{3m^2}{\sinh^2\left(mt\right)}, \quad m := \sqrt{\eta V_\infty/3}. \tag{3.3.24}$$

From (3.3.17) we see that the shear, the three-curvature and the energy density of the matter rapidly approach zero, just as in Wald's case. Also all components of the stress-energy tensor approach zero because of the dominant energy condition. Thus the universe isotropizes within one Hubble time of order $1/\sqrt{V_\infty} \sim 10^7\, t_{PL}$.

In the case of Bianchi IX models the scalar curvature ${}^{(3)}R$ may be positive and the above argument does not apply directly. In particular the function S is not bounded below from zero. However, as Wald [102] has pointed out, for some non-highly positively curved models, premature recollapse may be avoided provided a large positive cosmological constant compensates the $-\frac{1}{2}\,{}^{(3)}R$ term in (3.3.17). In our case of course, it is the potential $V\left(\varphi\right)$ which acts as a cosmological term. A similar argument for the Bianchi IX model was given by Kitada and Maeda [59] in the case of an exponential potential leading to power-law inflation. Although S is not positive, it might be non-negative initially. In that case, as long as $S\left(t\right) \geq 0$, the inequality (3.3.23) holds as in the previous case and an upper bound for S is given by (3.3.24). Otherwise S is bounded above from zero. Hence for general initial conditions we have an upper bound

$$S \leq \max\left\{0,\ 3m^2\sinh^{-2}\left(mt\right)\right\}. \tag{3.3.25}$$

An estimation of a (possible) lower bound for S derives from the fact that the largest positive value the spatial curvature can achieve is determined by the determinant of the three metric:

$$^{(3)}R_{\max} \propto h^{-1/3} := \exp\left(-2\alpha\right). \tag{3.3.26}$$

We obtain a lower bound for S in the following way. From (3.3.17) we have

$$\frac{1}{3}K^2 - \frac{1}{2}\dot{\varphi}^2 \geq 0.$$

Observe that for any $\lambda \in \left(0, \sqrt{2/3}\right)$, the above inequality implies that for $K > 0$

$$\frac{1}{3}K - \frac{1}{2}\lambda\,\dot{\varphi} \geq \frac{1}{3}K - \frac{1}{\sqrt{6}}\,|\dot{\varphi}| \geq 0. \tag{3.3.27}$$

Suppose that initially, *ie* at time t_i, we have $V > {}^{(3)}R_{\max}$. We claim that this inequality holds during the whole period of the effective regime of inflation. Define the function f by

$$f\left(t\right) := \ln\frac{V}{^{(3)}R_{\max}}, \quad t \in \left[t_i, t_f\right], \tag{3.3.28}$$

the initial valueof which is positive by the above assumption. Differentiating we find

$$\dot{f} = \frac{\dot{V}}{V} + 2\,\dot{\alpha} = 2\left[\frac{\exp\left(-\sqrt{\frac{2}{3}}\varphi\right)}{1-\exp\left(-\sqrt{\frac{2}{3}}\varphi\right)}\sqrt{\frac{2}{3}}\,\dot{\varphi} + \frac{1}{3}K\right] \tag{3.3.29}$$

since $K = 3\,\dot{\alpha}$, by (3.3.15). Since the coefficient of $\dot{\varphi}$ in the brackets is less than $1/\sqrt{6}$ during the effective regime of inflation, inequality (3.3.27) implies that $\dot{f} \geq 0$. Hence, if $K > 0$ and $f\left(t_i\right) > 0$ initially, then for $t \in \left[t_i, t_f\right]$ we have $f\left(t\right) \geq f\left(t_i\right)$ and our assertion folllows.

We are now in a position to estimate the required bound for S. Firstly, from (3.3.17) it is evident that

$$-\frac{1}{2}\,{}^{(3)}R_{\max} \leq S \tag{3.3.30}$$

and, since $V/2 > {}^{(3)}R_{\max}/2$, (3.3.17) again implies that

$$K \geq \sqrt{\frac{3}{2}V} > \sqrt{\frac{3}{2}\eta V_\infty}. \tag{3.3.31}$$

Remembering that $K = 3\,\dot{\alpha}$, from the last inequality and (3.3.26), we see that ${}^{(3)}R_{\max}$ decays faster than $\exp\left[-\sqrt{2\eta V_\infty/3}\left(t-t_i\right)\right]$. Hence inequality (3.3.30) is

$$-\frac{1}{2}\,{}^{(3)}R_{\max}\left(t_i\right)\exp\left[-\sqrt{\frac{2}{3}\eta V_\infty}\left(t-t_i\right)\right] \leq S. \tag{3.3.32}$$

Combining this lower bound for S with the upper bound (3.3.25), we conclude that S vanishes almost exponentially. From (3.3.17), $-2S \leq {}^{(3)}R \leq {}^{(3)}R_{\max}$, so ${}^{(3)}R$ damps to zero just as S and ${}^{(3)}R_{\max}$.

As in the case of the other Bianchi models, we see that, when the universe is initially on the plateau, the shear, the scalar three-curvature and all components of the stress-energy tensor approach zero almost exponentially fast with a time constant of order $\sim 1/\sqrt{V_\infty}$.[3]

We discussed inflation in the equivalent spacetime $(M, \widetilde{\mathbf{g}})$, but it is not obvious that the above attractor property is maintained in the original spacetime $(M, \mathbf{g})$. This is probably an unimportant question since there is much evidence that in most cases the rescaled metric $\widetilde{\mathbf{g}}$ is the real physical metric [28]. However, as Maeda has pointed out [71], since during inflation the scalar field changes very slowly and the two metrics are related by $\widetilde{\mathbf{g}} = \exp\left(\sqrt{2/3}\varphi\right)\mathbf{g}$, it is easily seen that inflation occurs in the original picture also.

In contrast to Wald's theorem, which is based on the existence of a true cosmological constant, the cosmological term in our case eventually vanishes. Therefore, it is not certain that the universe has enough time to evolve towards de Sitter spacetime during the evolution of the scalar field. In any consistent no-hair theorem one has to verify that the time necessary for isotropization is small compared to the time the field reaches the minimum of the potential.

The following argument shows that in our case the vacuum energy is not exhausted before the universe is completely isotropized. We imagine that at the beginning of inflation the universe is on the flat plateau of the potential. It is evident that the time $t_f - t_i$ needed for the scalar field to evolve from its initial value φ_i to a smaller value φ_f is smaller in the absence of damping than the corresponding time in the presence of damping. In the absence of damping the time $t_f - t_i$ is easily calculable because the equation of motion of the scalar field (3.3.19) can be integrated by elementary methods when $K = 0$. Taking for example the value φ_f to be such that $V\left(\varphi_f\right) \simeq \eta V_\infty$ one finds that $t_f - t_i$ is more than 65 times the time τ of isotropization. The presence of damping increases

[3]Actually, the time constant for isotropization in type IX is longer by $\sqrt{2}$ than in other types. The situation is similar to that encountered in Kitada and Maeda [59].

the time interval $t_f - t_i$ and a larger anisotropy damps more efficiently the slow rolling of the scalar field, thus producing more inflation. It follows that, when due account of damping is taken, the period of the effective regime of inflation is more than sufficient for both the complete isotropization of the universe and for the solution of the horizon problem.

We conclude with some comments about our proof as compared to the proof in the paper by Maeda [71] for Bianchi-type models except IX in vacuum. In that case ${}^{(3)}R \leq 0$ and $T_s = T_d = 0$. Moreover, as long as inflation continues, V remains less than V_∞. Hence Wald's method is directly applicable and equations (3.3.17) and (3.3.18) only (*ie*, without explicit use of the equation of motion of the scalar field) imply immediately

$$\begin{gathered}
\sqrt{3V_\infty} \leq K \leq \frac{\sqrt{3V_\infty}}{\tanh \alpha t} \\
\sigma^2 \leq \frac{V_\infty}{\sinh^2 \alpha t} \\
\dot{\varphi}^2 \leq \frac{2V_\infty}{\sinh^2 \alpha t},
\end{gathered} \tag{3.3.33}$$

where $\alpha = \sqrt{V_\infty/3}$. In our treatment we also show that the effective period of inflation is sufficient for the complete isotropization of the universe. This is again achieved using the equation of motion of the scalar field.

Chapter 4

Recollapse problems

A closed Friedmann universe is considered almost synonymous to a recollapsing universe. This is mainly due to our experience with the dust and radiation filled Friedmann models usually treated in textbooks. That this picture is misleading follows clearly from an example found by Barrow, Galloway and Tipler [8] according to which an expanding Friedmann model with spatial topology S^3 satisfying the weak, the strong, the dominant energy conditions and the generic condition may not recollapse. Thus the problem of recollapse of a closed universe to a second singularity is not trivial already in the Friedmann case.

There are many examples showing that under certain conditions on the matter content a closed universe recollapses. Several conjectures have been proposed concerning the recollapse problem (see [11] and references there). These conjectures usually assume the following general form:

Recollapse conjecture *All globally hyperbolic closed universes with spatial topology S^3 or $S^2 \times S^1$ and with stress-energy tensors satisfying the SEC and the PPC begin in an all-encompassing initial singularity and end in an all-encompassing final singularity.*

As mentioned above, the conjecture was found true in special cases; in certain spatially homogeneous cosmologies [66], in certain spherically symmetric spacetimes [108, 24] and in spacetimes admitting a constant mean-curvature foliation that possesses a maximal hypersurface [10, 87].

In this chapter we examine the closed universe recollapse conjecture in the context of a curvature squared higher order theory and give some plausibility arguments about the future asymptotic behavior of the closed

Friedmann universe in the conformal frame. In Section 4.1 we formulate the problem and compare with the standard closed Friedmann model with ordinary matter. In Section 4.2 we examine the conditions to be satisfied by the matter content in order that the classical global theorem for recollapse be applicable. In the last Section we summarize our results and illustrate some of the difficulties associated to the particular form of the self interacting potential.

4.1 The $k = 1$ FRW universe in the conformal frame

Starting with a Lagrangian $R+\beta R^2+L_{matter}$ and applying the conformal equivalence theorem, we obtain Einstein's equations with a scalar field having a potential V given by (see equations (3.3.4) and(3.3.6))

$$V(\varphi) = \frac{1}{8\beta}\left(1-e^{-\varphi}\right)^2 \equiv V_\infty\left(1-e^{-\varphi}\right)^2.$$

We consider the $k = 1$ FRW universe in the conformal frame. Ordinary matter is described by a perfect fluid with energy density ρ and pressure p. The Einstein equations become

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{1}{3}\left(\rho + \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \tag{4.1.1}$$

(Friedmann equation) and

$$2\frac{\ddot{a}}{a} = -\frac{1}{3}\left(\rho + 3p + 2\dot{\varphi}^2 - 2V\right) \tag{4.1.2}$$

(Raychaudhuri equation) and the equation of motion of the scalar field is

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + V'(\varphi) = 0. \tag{4.1.3}$$

From the last three equations one can find the conservation equation still holds:

$$\dot{\rho} + 3(\rho+p)\frac{\dot{a}}{a} = 0. \tag{4.1.4}$$

We assume an equation of state of the form $p = (\gamma-1)\rho$, $\gamma \in \left(\frac{2}{3}, 2\right)$.[1]

[1]The range of γ is chosen in accordance to the conditions for recollapse of Barrow, Galloway and Tipler (BGT) [8].

We see that the Friedmann $k = 1$ model in the conformal frame is formally obtained if we add to the matter content of the classical Friedmann universe a perfect fluid with energy density $E := \frac{1}{2} \dot{\varphi}^2 + V$ and pressure $q := \frac{1}{2} \dot{\varphi}^2 - V$ (see equations (3.3.9) and (3.3.10)). However, this fluid violates the SEC, *ie* $E + 3q = 2 \dot{\varphi}^2 - 2V$ may be negative for large values of the field. It is precisely this violation that leads to inflation in the very early universe as we saw in Chapter Three. Therefore we cannot conclude by the usual arguments [8] that the function a is convex downwards and the Theorem of BGT is not valid. Nevertheless we can imagine the following scenario. Suppose that at some time t_0 (today) the scalar field φ is negligible, *ie* it oscillates near the minimum of the potential. In an expanding universe the middle term in (4.1.3) damps the oscillations and ensures that the field does not grow. Hence we expect an almost classical Friedmannian evolution until the scale factor reaches its maximum value. However, once $\dot{a}$ changes sign, $3\left(\dot{a}/a\right)\dot{\varphi}$ in (4.1.3) acts as a driving force which forces the field φ to oscillate with larger and larger amplitude. It is then possible that φ climbs the hill of the potential and reaches the flat plateau. The repulsive effect of the cosmological term may change the sign of $\ddot{a}$ and so our model could avoid recollapse. An oscillating scale factor is then possible. The above intuitive ideas show that the system (4.1.1)–(4.1.3) may exhibit a great variety of different possible behaviours.

4.2 Conditions that ensure the existence of a maximal hypersurface

Before continuing our analysis we state some global results concerning the closed universe recollapse conjecture. The first is a theorem proven in [72]

Theorem *The length of every timelike curve in a spacetime* $(M, \mathbf{g})$ *is less than a constant* L *if the following conditions hold:*

(i) $(M, \mathbf{g})$ *is globally hyperbolic*

(ii) $Ric\,(u, u) \geq 0$ *for all timelike vectors* u

(iii) there exists a compact maximal hypersurface Σ

(iv) at least one of the tensors $n^r n^s n_{[a} R_{b]rs[c} n_{d]}$, K_{ab}, *or* $R_{ab} n^a n^b$ *is non-zero somewhere on* Σ, *where* n *is the unit vector field normal to* Σ *and* $K_{ab} = \nabla_a n_b$ *is the extrinsic curvature of* Σ.

We recall that a spacelike hypersurface Σ is said to be *maximal* if $\nabla_a n^a = 0$ on Σ, *ie* the expansion is zero. Global hyperbolicity is not a severe restriction because a very large class of known spacetimes are globally hyperbolic. In fact Penrose [85] has conjectured that all physically reasonable spacetimes must be globally hyperbolic. The meaning of the timelike convergence condition (ii) has been discussed in detail in Chapter Four. Condition (iv), says, that somewhere on Σ, either the gravitational forces are non-zero $\left(R_{ab} n^a n^b \neq 0\right)$, the tidal forces are non-zero $\left(n^r n^s n_{[a} R_{b]rs[c} n_{d]} \neq 0\right)$ or for vacuum spacetimes Σ is not a hypersurface of time symmetry $(K_{ab} \neq 0)$. Thus, for spacetimes having some kind of irregularity, condition (iv) seems physically plausible. The conclusion of the theorem, namely that the length of every timelike curve in the spacetime is less than L, has the interpretation that all timelike curves must begin at an initial singularity and terminate at a final singularity.

For completeness we state two other global results although we shall not make explicit use of them. The first says that the existence of a compact maximal hypersuface is also a necessary condition for a spacetime to have an initial and a final all-encompassing singularity. The second says roughly that the only globally hyperbolic spacetimes that *can have* a maximal hypersurface are those with spatial topology S^3 or $S^2 \times S^1$ (see [11] for a discussion).

We turn now to the problem of recollapse of the Friedmann universe in a $R + R^2 + L_{matter}$ theory in the conformal frame. By the above theorem, it is not necessary to examine the detailed behavior of the solution of (4.2.1)–(4.2.3) for large values of t: it suffices to check if the scale factor has a maximum. If this is the case, then a Friedmann universe with spatial topology S^3 possesses a compact maximal hypersurface. Hence, if the other conditions of the theorem are also satisfied, this universe recollapses.

In the following we use a new variable α defined by

$$\dot{\alpha} = \frac{\dot{a}}{a}, \quad \dot{\alpha} \in (-\infty, +\infty)$$

In terms of α equations (4.1.1)–(4.1.3) are

$$\dot{\alpha}^2 + e^{-2\alpha} = \frac{1}{3}\left(\rho + \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \tag{4.2.1}$$

$$2\ddot{\alpha} + 2\dot{\alpha}^2 = -\frac{1}{3}\left(\rho + 3p + 2\dot{\varphi}^2 - 2V\right) \tag{4.2.2}$$

$$\ddot{\varphi} + 3\dot{\alpha}\dot{\varphi} + V'(\varphi) = 0. \tag{4.2.3}$$

The conservation equation

$$\dot{\rho} + 3(\rho + p)\dot{\alpha} = 0$$

can be integrated to give

$$\rho = \mathrm{const} \times \exp(-3\gamma\alpha). \tag{4.2.4}$$

From (4.2.3) we obtain (compare to 3.3.20))

$$\dot{E} = -3\dot{\alpha}\dot{\varphi}^2 \tag{4.2.5}$$

which implies that in an expanding universe,

$$\dot{E} \leq 0, \tag{4.2.6}$$

ie, the field looses energy. Assume that at time t_0 (now) the values of φ and $\dot{\varphi}$ are very small in the sense that

$$E_0 := \frac{1}{2}\dot{\varphi}_0^2 + V_0 \ll \rho_0 \quad \text{and} \quad 2\dot{\varphi}_0^2 - 2V_0 \ll \rho_0 + 3p_0 \tag{4.2.7}$$

so that the total stress-energy tensor satisfies the SEC. This is a plausible assumption since the scalar field is unobservable in the present universe. Equation (4.2.6) implies that there exists a time interval $[t_0, t_0 + T]$ such that φ and $\dot{\varphi}$ remain small and in this interval the universe evolves according to the standard $k = 1$ FRW model. The scale factor in the standard $k = 1$ FRW model (with no scalar field and $p = (\gamma - 1)\rho$, $\gamma \in (2/3, 2])$ reaches a maximum at some time $t_{\max}$ and decreases to zero. It may happen that T is large enough and our model also has a time of maximum expansion. However, it is possible that after the time $t_0 + T$ the energy density $\frac{1}{2}\dot{\varphi}^2 + V$ and pressure $\frac{1}{2}\dot{\varphi}^2 - V$ of the scalar field dominate over the density ρ and pressure p of ordinary matter. This is possible in an expanding universe with $t_{\max} \gg t_0 + T$ since ρ decreases as $\sim \exp(-3\gamma\alpha)$. In that case, since the stress-energy tensor of the scalar field violates both the SEC and the PPC, the universe may have not a time of maximum expansion.

The above discussion suggests that a detailed analysis of the time dependence of the energy density of the scalar field is necessary, that is, we have to examine if the conditions (4.2.7) remain true for all $t \geq t_0$. To this end we write (4.2.3) (in first order approximation) as

$$\ddot{\varphi} + 3\,\dot{\alpha}\dot{\varphi} + m^2\varphi = 0 \tag{4.2.8}$$

where $m^2 := 2V_\infty$. This is the equation of motion of an harmonic oscillator with a variable damping factor $3\,\dot{\alpha}$. For a slowly varying function α this equation can be solved using the Kryloff-Bogoliuboff [62] approximation. We find that the amplitude of the scalar field varies as $\sim \exp\left(-\frac{3}{2}\alpha(t)\right)$. Since the amplitude of $\dot{\varphi}$ has the same time dependence and in our approximation $E = \frac{1}{2}\,\dot{\varphi}^2 + \frac{1}{2}m^2\varphi^2$, it follows that

$$E \sim \exp(-3\alpha). \tag{4.2.9}$$

Comparing this with the time dependence of the density $\rho \sim \exp(-3\gamma\alpha)$ we arrive at the following results:

- If $\gamma \leq 1$ the initial conditions (4.2.7) imply that for $t \geq t_0$

$$\begin{aligned} &E = \tfrac{1}{2}\,\dot{\varphi}^2 + V \ll \rho \quad \text{and} \\ &2\,\dot{\varphi}^2 - 2V \ll \rho + 3p \end{aligned}$$

  Hence the universe follows the classical Friedmannian evolution and has a time of maximum expansion. By the preceding discussion this is equivalent to the existence of a (compact) maximal hypersurface. It is easy to verify that the other conditions of the above theorem are also satisfied: (i) Global hyperbolicity follows from the fact that every S_t^3 of constant t in $\mathbb{R} \times S^3$ is a Cauchy surface. (ii) The TCC is implied by the SEC on the total stress-energy tensor. Taking for example for u in (ii) of Theorem 2 to be the unit normal to S_t^3 vector field n, we have $Ric(n,n) = \frac{1}{2}\left(\rho + 3p + 2\,\dot{\varphi}^2 - 2V\right) \geq 0$. The same argument shows that condition (iv) is also satisfied.

Therefore, if $\gamma \leq 1$, the universe recollapses. The case $\gamma = 1$ is particularly interesting since it corresponds to a dust-filled universe which perfectly approximates our Universe.

- If $\gamma > 1$ there are two possibilities. Either $t_{\max}$ is far in the future or $t_{\max} \in [t_0, t_0 + T]$ (remember that $t_{\max}$ is the time of maximum expansion of the universe filled with *ordinary matter*, *ie* without a scalar field). In the former case, since ρ decreases faster than the energy density E of the scalar field, there exists some time t_1 such that, for $t \geq t_1$, E dominates over ρ. Hence this model may have not a time of maximum expansion. In the later case, it may happen that the scale factor reaches a maximum value before the energy density and pressure of the scalar field exceed the values of the corresponding quantities ρ and p (this is most probable when the pressure p is small compared to the energy density ρ, *ie* when γ is not much larger than unity). Which particular scenario will actually take place depends on the detailed specification of the parameters of the model and a fine tuning of the initial conditions.

4.3 Summary

For a curvature squared gravity theory in the conformal frame the Friedmann $k = 1$ universe filled with dust (or more generally with a perfect fluid with $p = (\gamma - 1)\,\rho, \;\; \gamma \leq 1$) recollapses. The case $\gamma > 1$ cannot be handled by the above procedure. However, in the physically realistic case, where, γ is slightly larger than 1, it seems that the universe cannot avoid recollapse. Furthermore, a strict result for $\gamma > 1$, at least for $1 < \gamma \leq 4/3$, could possibly be obtained by the detailed analysis of the system (4.2.1)–(4.2.3) either by classical or by global methods.

We mention two successful techniques that can be used in the case of a scalar field as the only matter source. The first is the phase portrait analysis of the Friedmann model with an exponential potential given by Halliwell [48] (see also [59]). This is possible since, in that case, the potential function V has the nice property of being proportional to the derivative V' and, therefore, the resulting autonomous system becomes two-dimensional. Our case subtler, since the corresponding dynamical system turns out to be three-dimensional. The second is that during the inflationary regime one can treat the field φ as a time parameter (see [13]). In our case this is not possible, since the field oscillates near the minimum of the potential during the quasi-Friedmann period and, therefore, $\dot{\varphi}$ changes rapidly sign.

The possibility to generalize the above results in the case of a Bianchi type IX cosmology, might also be the subject of further research. A possible approach is along the lines of the theorem of Lin and Wald [66], where it seems that the shear is mainly responsible for recollapse of the Bianchi IX model.

Chapter 5

Conclusion

In the first part of this thesis we analyzed the variational structure of arbitrary nonlinear Lagrangian theories of gravity. In particular we proved that the only consistent way to generalize higher order gravity theories in the framework of non-Riemannian geometries is the constrained Palatini variation. We applied this result to $f(R)$ theories in Weyl geometry and studied the conformal structure of the corresponding field equations. It turns out that the Palatini variation is a degenerate case corresponding to a particular gauge and in the limit when the Weyl vector field vanishes one obtains the conformal equivalence theorem mentioned in the Introduction. Furthermore the field equations exhibit a rich structure with possibly interesting properties that depend upon particular choices of the source term, *ie* the Weyl vector field. We think that further generalization to the case of non-symmetric connections is possible along the lines of the theorem in Chapter Two. It should be interesting to consider applications in cosmological models. Such a program could start by analyzing the properties of Friedmann cosmologies, find their singularity structure and examine the possibility of inflation. The next step could be to consider the past and future asymptotic states of Bianchi cosmologies that is, to examine isotropization and recollapse conjectures in such universes. This will not be as straightforward as in the Riemannian case because of the presence of the source term M_{ab}.

The second part of this thesis was mainly devoted to the isotropization problem in higher order gravity cosmologies. Our first result was the proof of the cosmic no-hair conjecture for homogeneous cosmologies in the context of the curvature squared Lagrangian with matter. In particular we proved a no-hair theorem to the effect that the Bianchi type IX universe model isotropizes during the very early epoch provided that initially the scalar three-curvature does not exceed the potential of the scalar field associated to the conformal transformation. The fact that the cosmic no-hair conjecture is probably true for a large class of gravity theories, at least for homogeneous cosmologies, can be interpreted as a generic property shared by these theories. We believe that our proof of the cosmic no–hair conjecture could be extended with adjustments to include the class of tilted Bianchi models. In fact such a demonstration could amount to a first test of this conjecture in cases of some inhomogeneity. An analysis along these lines might be more tractable than, say, attacking directly a genuine inhomogeneous case such as, for instance, that of G_2 cosmologies wherein the dynamics is described by systems of partial differential equations. However, the generalization to arbitrary spacetimes seems mathematically intractable at the moment.

The generalization of the (first) Collins-Hawking isotropization theorem, for a rather large class of $f(R)$ theories, was our second result. Global results in higher order gravity theories, such as the usual ones in general relativity, are very difficult to obtain in the Jordan frame, mainly because the positivity properties of the Ricci tensor cannot easily tested in these theories. In particular the Raychaudhuri equation, which proved to be an indispensable tool for the study of singularities in general relativity, does not give direct information in higher order gravity. Our derivation of the so-called Raychaudhuri system permits the transfer to $f(R)$ theories some known results valid in general relativity.

The closed universe recollapse conjecture in the curvature-squared gravity theory was discussed and sufficient conditions for recollapse of the closed Friedmann model were given. It should be interesting to consider the more general case, namely the Bianchi type IX, in this framework and compare with the results obtained in [33].

Appendix A

Raychaudhuri's equation

Consider a smooth congruence of timelike geodesics in a spacetime $(M, \mathbf{g})$. The corresponding tangent vector field $\mathbf{n}$ is normalized to unit length, $\langle \mathbf{n}, \mathbf{n} \rangle = -1$. This means that the geodesics are parametrized by proper time t and $\mathbf{n} = \partial/\partial t$. We define the *spatial metric* $\mathbf{h}$ by

$$h_{ab} = g_{ab} + n_a n_b. \tag{A.0.1}$$

Note that $h_a{}^b n_b = h^b{}_a n^a = 0$ so that $h^a{}_b = g^{ac} h_{cb}$ can be regarded as the projection operator onto the subspace of the tangent space perpendicular to $\mathbf{n}$. In the following we are interested on the covariant derivative of $\mathbf{n}$. Its geometric meaning will become clear at the end of this Appendix.

We define the *expansion,* θ, *shear,* σ, and *rotation,* ω, of the congruence by

$$\theta = h^{ab} \nabla_b n_a = \nabla_a n^a \tag{A.0.2}$$

$$\sigma_{ab} = \nabla_{(b} n_{a)} - \frac{1}{3} \theta h_{ab} \tag{A.0.3}$$

$$\omega_{ab} = \nabla_{[b} n_{a]}. \tag{A.0.4}$$

The tensor fields σ_{ab} and ω_{ab} are purely spatial in the sense that $\sigma_{ab} n^b = \omega_{ab} n^b = 0$ and σ_{ab} is traceless. If the stress-energy tensor of the matter fields is of the form of a fluid, then θ, σ, and ω, are *not* the expansion, shear and the rotation of the fluid unless the fluid happens to be moving along the geodesics.

The covariant derivative of $\mathbf{n}$ can now be expressed as

$$\nabla_b n_a = n_{a;b} = \frac{1}{3} \theta h_{ab} + \sigma_{ab} + \omega_{ab}. \tag{A.0.5}$$

This can be verified by direct substitution from the previous defining equations (A.0.2) – (A.0.4).[1]

From the definition of the curvature tensor and the geodesic equation $n^a \nabla_a n^b = 0$ we have

$$\begin{aligned} n^c \nabla_c \nabla_b n_a &= n^c \nabla_b \nabla_c n_a + R_{adcb} n^d n^c \\ &= \nabla_b \left(n^c \nabla_c n_a \right) - \left(\nabla_b n^c \right) \left(\nabla_c n_a \right) + R_{adcb} n^d n^c \\ &= - \left(\nabla_b n^c \right) \left(\nabla_c n_a \right) + R_{adcb} n^d n^c . \end{aligned}$$

Taking the trace of the last equation we obtain

$$\frac{d\theta}{dt} := n^c \nabla_c \left(\nabla_d n^d \right) = - \left(\nabla_d n^c \right) \left(\nabla_c n^d \right) - R_{cd} n^c n^d$$

and using (A.0.5) we obtain after some manipulation that

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab} n^a n^b . \tag{A.0.6}$$

This equation is known as the Raychaudhuri equation and plays an important role in the proof of the singularity theorems of general relativity.

A.1 Extrinsic curvature

Consider now the case that the spacetime $(M, \mathbf{g})$ is globally hyperbolic and the congruence of the timelike geodesics is normal to a spacelike hypersurface Σ. In every point p of Σ the unit normal to Σ at p equals the (unit) tangent vector $\mathbf{n}$ of the geodesic passing through p. Then the *induced metric tensor* $\mathbf{h}$ on Σ coincides with the previously defined 'spatial metric' $h_{ab} = g_{ab} + n_a n_b$, (A.0.1). For the same reason the covariant derivative of $\mathbf{n}$ evaluated on Σ coincides with the *extrinsic curvature* K_{ab} of Σ, viz.

$$K_{ab} = \nabla_a n_b . \tag{A.1.1}$$

Of course the tensor field $\mathbf{K}$ is purely spatial (see footnote on page 72). Since the congruence is hypersurface orthogonal,[2] we have $\omega_{ab} = 0$ which

[1] In HE there is a projection $h_a{}^b$ for every index of the tensor field $n_{a;b}$ in the definitions of the shear and the rotation. Our definitions differ from those in HE because $n_{a;b}$ is purely spatial, $n_{a;b} n^a = n_{a;b} n^b = 0$. This is due to the fact that n is associated with a congruence of geodesics, not merely a congruence of timelike curves.

[2] A necessery and suficient condition that n be hypersurface orthogonal is $n_{[a} \nabla_b n_{c]} = 0$. See for example Wald [101], appendix B, p 436.

implies that the extrinsic curvature tensor field is symmetric *ie* $K_{ab} = K_{ba}$. Hence taking the Lie derivative of the metric with respect to $\mathbf{n}$ we find

$$\begin{aligned} K_{ab} &= \tfrac{1}{2}\mathfrak{L}_{\mathbf{n}} g_{ab} \\ &= \tfrac{1}{2}\mathfrak{L}_{\mathbf{n}} \left(h_{ab} - n_a n_b\right) \\ &= \tfrac{1}{2}\mathfrak{L}_{\mathbf{n}} h_{ab}, \end{aligned} \tag{A.1.2}$$

where the geodesic equation was used. If a coordinate system adapted to $\mathbf{n}$ is used, then the components of the extrinsic curvature in these coordinates are

$$K_{\mu\nu} = \frac{1}{2}\frac{\partial h_{\mu\nu}}{\partial t}. \tag{A.1.3}$$

The trace K of the extrinsic curvature is defined as

$$K := K^a{}_a = h^{ab} K_{ab} = \theta \tag{A.1.4}$$

so that K is equal to the mean expansion θ of the geodesic congruence orthogonal to Σ.

One has the following geometric interpretation of K [90]. Assume Σ to be a compact submanifold with boundary (otherwise, take a compact subset of Σ). For every p in Σ denote by γ_p the geodesic passing through p, *ie* $\gamma_p : [0, \varepsilon] \to M$ is a future-directed geodesic orthogonal to Σ and satisfying $\gamma_p(0) = p$, with tangent vector field $\mathbf{n}$. For all $t \in [0, \varepsilon]$, define $\Sigma_t := \left\{\gamma_p(t) : p \in \Sigma\right\}$, that is, Σ_t is the set of all points of Σ moved along each geodesic a parametric distance t. If $V(t)$ denotes the Riemannian volume of Σ_t, then it can be shown that

$$V'(0) = \int_{\Sigma} K \Omega_{\Sigma}, \tag{A.1.5}$$

where Ω_{Σ} is the Riemannian volume element of Σ. Thus $K > 0$ roughly means that the future-directed geodesics orthogonal to Σ are, on the average, spreading out near Σ so as to increase the volume of Σ.

Appendix B

Homogeneous cosmologies

The spacetimes we deal with in this work are with a few exceptions spatially homogeneous. A spacetime $(M, \mathbf{g})$ is said to be *spatially homogeneous* if there exists a one-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime such that for each t and for each $p, q \in \Sigma_t$ there exists an isometry of M which takes p to q. Homogeneous cosmologies have been studied intensively over the past years (for reviews see Ryan and Shepley [88], MacCallum [69, 70]). One of the main reasons is that all possible geometries of the spacelike hypersurfaces fall into one of ten classes. Another important reason is that Einstein's equations reduce to a system of ordinary differential equations. In fact because of the spatial symmetry only time variations are non-trivial.

The set of all isometries of a Riemannian manifold forms a Lie group G and the set of the Killing vector fields (that is, the set of infinitesimal generators of the isometries) constitutes the associated Lie algebra, with product the commutator. In our case $\dim G = \dim \Sigma_t = 3$ provided that G acts *simply transitively* on each Σ_t, that is, for every p, q in Σ_t there exists a *unique isometry* $\in G$ sending p to q. The algebraic structure of the group G can be described in terms of the Lie algebra since, if v, w are Killing vector fields, they satisfy

$$[v, w]^a = -C^a{}_{bc} v^b w^c, \tag{B.0.1}$$

where $C^a{}_{bc}$ are the *structure constants* of the Lie group. It follows immediately from the definition that

$$C^a{}_{bc} = -C^a{}_{cb} \tag{B.0.2}$$

and from the Jacobi identity for commutators that

$$C^e{}_{d[a}C^d{}_{bc]} = 0. \tag{B.0.3}$$

These two equations lead to all three-dimensional Lie groups, or equivalently, all possible sets of structure constants which satisfy (B.0.2) and (B.0.3). Bianchi was the first to classify all three-dimensional Lie groups into nine types. A slightly different version of this classification can be obtained in the following way (see Ellis and MacCallum [41]). The tensor field $C^c{}_{ab}$ can be decomposed as

$$C^c{}_{ab} = M^{cd}\epsilon_{dab} + \delta^c{}_{[a}A_{b]}, \tag{B.0.4}$$

where $\epsilon_{abc} = \epsilon_{[abc]}$ is a three form on the Lie algebra, $M^{cd} = M^{dc}$ and A_b is a 'dual' vector. We can solve for M^{ab} and A_b, taking $A_b = C^a{}_{ba}$ and $M^{ab} = \frac{1}{2}C^{(a}{}_{cd}\epsilon^{b)cd}$. Inserting (B.0.4) into the Jacobi identity (B.0.3) yields

$$M^{ab}A_b = 0. \tag{B.0.5}$$

Therefore the problem of finding all three-dimensional Lie groups is reduced to determining all dual vectors A_b and all symmetric tensors M^{ab} satisfying (B.0.5). If $A_b = 0$ (class A), there exist six Lie algebras determined by the rank and signature of M^{ab}. If $A_b \neq 0$ (class B), equation (B.0.5) implies that rank$M$ cannot be greater than two. Hence, in this case there exists four possibilities for the rank and signature of M^{ab}. These ten combinations are tabulated (see *eg* in Landau and Lifshitz [63], MacCallum [70]) and called the Bianchi models. For example the Bianchi type-IX model is determined by $A_b = 0$, rank$M = 3$, signature$M = (+++)$. One can find explicit formulas for $C^c{}_{ab}$ in different bases in Ryan and Shepley [88]. Useful formulas for the Ricci tensor and Einstein equations in terms of the structure constants and the spatial metric are also given in Ryan and Shepley [88].

The metric of a spatially homogeneous spacetime is

$$g_{ab} = -n_a n_b + h_{ab}, \tag{B.0.6}$$

where **h** is the three-metric of the spatial slices and $\mathbf{n} = \partial/\partial t$ is a unit timelike vector field, orthogonal to the homogeneous hypersurfaces. The vector field **n** defines the time coordinate of the spacetime. There are many ways to put the metric in a useful form [69]. For example the spatial coordinates can be chosen as follows. We consider one homogeneous hypersurface Σ_0 and choose a basis of one-forms $\omega^1, \omega^2, \omega^3$ which are preserved under the isometries, that is, have zero Lie derivative with respect to the Killing vector fields. It follows that each one-form ω^i (the index i labels the one-form) satisfies (see for example Wald [101] Section 7.2)

$$\nabla_{[a}\omega^i_{b]} = -\frac{1}{2}C^c{}_{ab}\omega^i_c \tag{B.0.7}$$

with $C^c{}_{ab}$ the structure constants of the Lie group of isometries of the spacelike hypersurfaces. Then the (invariant) spatial metric can be written

$$h_{ab} = h_{ij}\omega^i_a\omega^j_b, \quad i,j = 1,2,3, \tag{B.0.8}$$

where the components h_{ij} are constant on Σ_0.

To construct the full metric we consider for $p \in \Sigma_0$ the unit normal vector $\mathbf{n}_p$ to Σ_0 at p (we used the symbol **n** to denote an arbitrary timelike vector field orthogonal to the homogeneous hypersurfaces for reasons that will soon become clear). Denote by γ_p the geodesic determined by $(p, \mathbf{n}_p)$. Then γ_p will be orthogonal to all the spatial hypersurfaces it intersects, because the tangent to γ_p remains orthogonal to all the spatial Killing vector fields (see for example O'Neill [82], ch.9, lemma 26). We label the other hypersurfaces by the proper time t of the intersection of γ_p with the hypersurface. Hence $t = const$ on each Σ_t. Then the vector field **n** defined by $n^a = -\nabla^a t$ will be everywhere orthogonal to each Σ_t and the integral curves of **n** are all geodesics since this is true along γ_p and hence is true everywhere on each Σ_t by spatial homogeneity. Now we 'Lie transport' the ω^i defined on Σ_0 throughout the spacetime along **n**, *ie*

$$\mathfrak{L}_{\mathbf{n}}\omega^i = 0,$$

which implies that $\omega^i_a n^a = 0$ everywhere. We conclude that the metric (B.0.6) takes the form

$$g_{ab} = -\nabla_a t\nabla_b t + h_{ij}(t)\,\omega^i_a\omega^j_b, \tag{B.0.9}$$

or equivalently,

$$ds^2 = -dt^2 + h_{ij}(t)\,\omega^i\omega^j, \quad i,j = 1,2,3. \tag{B.0.10}$$

The property of homogeneous spacetimes mentioned at the beginning of this Appendix is now clear, namely that the Einstein equations become ordinary differential equations with respect to time.

We note by ${}^{(3)}R$ the scalar curvature of the spacelike hypersurface which we think of as a Riemannian three-manifold with metric $\mathbf{h}$. In what follows we make repeated use of a property of the scalar spatial curvature ${}^{(3)}R$, namely that ${}^{(3)}R$ is nonpositive in all Bianchi models except type-IX. To prove it, we write the scalar curvature ${}^{(3)}R$ in terms of the structure constants $C^a{}_{bc}$ of the Lie algebra of the symmetry group of the homogeneous hypersurface (see [63] or [88])

$$ {}^{(3)}R = -C^a{}_{ab}C^c{}_c{}^b + \frac{1}{2}C^a{}_{bc}C^c{}_a{}^b - \frac{1}{4}C_{abc}C^{abc}. \tag{B.0.11} $$

All indices are lowered and raised with the spatial metric, h_{ab}, and its inverse h^{ab}. A rather lengthy calculation gives for ${}^{(3)}R$ (by substitution of (B.0.4) into (B.0.11) and using (B.0.5))

$$ {}^{(3)}R = -\frac{3}{2}A_bA^b - h^{-1}\left(M^{ab}M_{ab} - \frac{1}{2}M^2\right), \tag{B.0.12} $$

where h is the determinant of h_{ab}, that is, $h^{-1} = \epsilon_{abc}\epsilon_{mnr}h^{am}h^{bn}h^{cr}$. From (B.0.12) it follows immediately that, if ${}^{(3)}R$ is positive then necessarily $M^{ab}M_{ab} < \frac{1}{2}M^2$, but then M^{ab} must be definite (positive or negative) as can be verified by considering a coordinate system where the tensor M^{ab} is diagonal. In this case (B.0.5) implies that $A_b = 0$. A look at the Bianchi classification shows that the combination $A_b = 0$ and rank$M = 3$ corresponds to the type-IX model. What we have shown is that in all Bianchi models except type-IX the three-scalar curvature is nonpositive, *ie*

$$ {}^{(3)}R \leq 0 \, . \tag{B.0.13} $$

This ends the necessary geometric notions which will be used in the development of this thesis.

Appendix C

Energy conditions

The actual form of the stress-energy tensor of the universe is very complicated since a large number of different matter fields contribute to form it. Therefore it is hopeless to try to describe the precise form of the stress-energy tensor. However, there are some inequalities which it is physically reasonable to assume for the stress-energy tensor. In many circumstances these inequalities are sufficient to prove via the field equations many important global results, for example the singularity theorems. In this Appendix we discuss these inequalities, usually referred to as energy conditions.

If $\mathbf{n}$ is a unit timelike vector field, the quantity $T_{ab}n^a n^b$ is the energy density as measured by an observer whose 4-velocity is $\mathbf{n}$. For all known forms of matter this energy density is non-negative and therefore we impose that

$$T_{ab}u^a u^b \geq 0 \tag{C.0.1}$$

for all timelike vectors u. This condition is known as the *weak energy condition* (WEC). An other assumption usually accepted is the *strong energy condition* (SEC) which states that

$$T_{ab}n^a n^b + \frac{1}{2}T \geq 0 \tag{C.0.2}$$

for all unit timelike vectors n. In fact it seems reasonable that the matter stresses will not be so large as to make the right-hand side of (C.0.2) negative.

One can see that, if the right-hand side of the Raychaudhuri equation (A.0.6) is negative, the expansion θ of a congruence of timelike geodesics decreases along a geodesic. We pay therefore attention to the sign of the $R_{ab}n^a n^b$ term on the right-hand side of the Raychaudhuri equation. The expansion θ of a congruence of timelike geodesics decreases if

$$R_{ab}n^a n^b \geq 0 \text{ for any unit timelike vector } n. \tag{C.0.3}$$

Inequality (C.0.3) is what Hawking and Ellis [50] call the *timelike convergence condition* (TCC). By the Einstein equations the term $R_{ab}n^a n^b$ can be written as

$$R_{ab}n^a n^b = \left(T_{ab} - \frac{1}{2}T g_{ab}\right) n^a n^b = \left(T_{ab}n^a n^b + \frac{1}{2}T\right). \tag{C.0.4}$$

Therefore in general relativity the timelike convergence condition is equivalent to the strong energy condition. However, this is not generally true for an arbitrary $f(R)$ gravity theory.

Finally the *dominant energy condition* states that

$$T_{ab}u^a u^b \geq 0 \tag{C.0.5}$$

and $T^a{}_b u^b$ is a non-spacelike vector for all timelike vectors u. In particular the dominant energy condition implies that

$$|T_{\mu\nu}| \leq T_{00}, \tag{C.0.6}$$

where $T_{\mu\nu}$ are the components of T_{ab} in any orthonormal basis with n as the timelike element of this basis.

To be convinced that the above conditions are reasonable assumptions, let us see what they imply for a diagonalizable stress-energy tensor $T_{\mu\nu} = diag\,(\rho, p_1, p_2, p_3)$. This is for example the case of a perfect fluid with stress-energy tensor $T_{\mu\nu} = diag\,(\rho, p, p, p)$. It is easy to see that the energy conditions take the form

$$\begin{aligned} &\rho \geq 0 \text{ and } \rho + p_i \geq 0 \quad (i = 1,2,3) \quad \text{(WEC)} \\ &\rho + p_1 + p_2 + p_3 \geq 0 \text{ and } \rho + p_i \geq 0 \quad (i = 1,2,3) \quad \text{(SEC)} \\ &\rho \geq |p_i| \quad (i = 1,2,3) \quad \text{(DEC)}. \end{aligned} \tag{C.0.7}$$

For further discussion see for example HE [50].

Appendix D

Conformal transformations

Consider the transformation[1]

$$\widetilde{\mathbf{g}} = \Omega^2 \mathbf{g}, \tag{D.0.1}$$

where Ω^2 is a smooth strictly positive function. Loosely speaking, a transformation of the form (D.0.1) represents a rescaling of the metric in each spacetime point. Since the angle between any two directions remains the same, conformally related spacetimes have identical causal structure: a vector is timelike, null or spacelike with respect to the metric $\mathbf{g}$ if and only if it has the same property with respect to the metric $\widetilde{\mathbf{g}}$. However, conformally related spacetimes may have different topologies. Consider for example the metric of Minkowski spacetime $(\mathbb{R}^4, \boldsymbol{\eta})$

$$ds^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2$$

In spherical coordinates (t, r, θ, ϕ) this metric becomes

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{D.0.2}$$

Using null coordinates u and v defined by $v = t + r$, $u = t - r$, we see that the metric is

$$ds^2 = -dudv + \frac{1}{4}(v-u)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{D.0.3}$$

[1]Strictly speaking the metrics in the transformation (D.0.1) are said to be *conformally related,* while a *conformal transformation* is a diffeomorphism $f : (M, \mathbf{g}) \to (M, \widetilde{\mathbf{g}})$ such that $f^*\mathbf{g} = \Omega^2\mathbf{g}$; see *eg* [81] for a discussion.

where $-\infty < u < +\infty$, $-\infty < v < +\infty$. With the definition of new coordinates p, q by $\tan p = v$, $\tan q = u$ where $-\pi/2 < p < \pi/2$, $-\pi/2 < q < \pi/2$, the metric (D.0.3) becomes

$$ds^2 = \frac{1}{4}\frac{1}{\cos^2 p}\frac{1}{\cos^2 q}\left[-4dpdq + \sin^2(p-q)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \quad \text{(D.0.4)}$$

This metric can be reduced to a more usual form by defining $T = p + q$, $R = p - q$, where

$$-\pi < T + R < \pi, \quad -\pi < T - R < \pi, \quad R \geq 0. \quad \text{(D.0.5)}$$

The result is

$$ds^2 = \frac{1}{4}\frac{1}{\cos^2 p}\frac{1}{\cos^2 q}\left[-dT^2 + dR^2 + \sin^2 R\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \quad \text{(D.0.6)}$$

Now the metric $\widetilde{\mathbf{g}}$ defined by $ds^2 = -dT^2 + dR^2 + \sin^2 R\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ is the natural Lorenz metric on $S^3 \times \mathbb{R}$, known as the Einstein static universe, except that the coordinate ranges are restricted by (D.0.5). We see that *Minkowski spacetime* $(\mathbb{R}^4, \boldsymbol{\eta})$ *is conformally related to that part of the Einstein static universe* $(S^3 \times \mathbb{R}, \boldsymbol{\eta})$ *defined by the coordinate system (D.0.5).* The conformal factor is $\Omega = 2\cos p\cos q$. Thus, the whole Minkowski spacetime has been conformally compactified into this finite region. The conformal compactification procedure has been widely used by Penrose (see HE [50] for a discussion and applications).

We briefly summarize the relations between geometric quantities defined on the two conformally related spacetimes $(M, \mathbf{g})$ and $(M, \widetilde{\mathbf{g}})$. Let ∇ denote the covariant derivative operator compatible with the metric $\mathbf{g}$ and let $\widetilde{\nabla}$ denote the covariant derivative operator associated with the metric $\widetilde{\mathbf{g}}$. The relation between $\widetilde{\nabla}$ and ∇ is given by the equation [105, 101]

$$\widetilde{\nabla}_a\omega_b = \nabla_a\omega_b - C^c_{ab}\omega_c, \quad \text{(D.0.7)}$$

where

$$C^c_{ab} = \frac{1}{2}\widetilde{g}^{cm}\left(\nabla_a\widetilde{g}_{bm} + \nabla_b\widetilde{g}_{am} - \nabla_m\widetilde{g}_{ab}\right). \quad \text{(D.0.8)}$$

Since $\nabla_a\widetilde{g}_{bc} = \nabla_a\left(\Omega^2 g_{bc}\right) = 2\Omega g_{bc}\nabla_a\Omega$, we obtain for C^c_{ab}

$$C^c_{ab} = \delta^c_a\nabla_b\ln\Omega + \delta^c_b\nabla_a\ln\Omega - g_{ab}g^{cm}\nabla_m\ln\Omega. \quad \text{(D.0.9)}$$

Hence we have a relation between the two connections $\widetilde{\nabla}$ and ∇ in terms of the initial metric $\mathbf{g}$ and the conformal factor Ω. Therefore all interesting geometric objects defined on $(M, \mathbf{g})$ and $(M, \widetilde{\mathbf{g}})$ in terms of the connection, namely geodesics, curvature etc. can be related via the initial metric $\mathbf{g}$ and the conformal factor Ω.

As an example consider an affinely parametrized geodesic γ in $(M, \mathbf{g})$ with tangent vector field n. Then $\mathbf{n}$ satisfies $\nabla_{\mathbf{n}}\mathbf{n} = 0$ or, in index notation, $n^a \nabla_a n^b = 0$. What is the corresponding relation on $(M, \widetilde{\mathbf{g}})$? By eqs. (D.0.7) and (D.0.9) we have

$$n^a \widetilde{\nabla}_a n^b = n^a \nabla_a n^b + n^a C^b_{ac} n^c = 2n^b n^c \nabla_c \ln\Omega - (n_a n^a)\, g^{bm} \nabla_m \ln\Omega. \tag{D.0.10}$$

Thus in general γ fails to be a geodesic with respect to $\widetilde{\nabla}$. However, in the case of a null geodesic, $n_a n^a = 0$, equation (D.0.10) shows that γ is a *pregeodesic* [82], that is, the tangent $\mathbf{n}$ satisfies

$$\widetilde{\nabla}_{\mathbf{n}}\mathbf{n} = f\mathbf{n}, \tag{D.0.11}$$

where the function f is $f = 2\nabla_{\mathbf{n}} \ln\Omega$. A pregeodesic can allways reparametrized to be a geodesic. Hence null geodesics are conformally invariant.

The components of the Riemann curvature tensor in $(M, \widetilde{\mathbf{g}})$ may be found by the usual formula in terms of the connection coefficients $\widetilde{\Gamma}^a_{bc}$ which are related to the Γ^a_{bc} by $\widetilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + C^a_{bc}$. However, we can proceed starting from the definition of the Riemann tensor

$$\nabla_c \nabla_d Z^a - \nabla_d \nabla_c Z^a = R^a{}_{bcd} Z^b$$

and use eqs. (D.0.7) and (D.0.9) to find

$$\begin{aligned} \widetilde{R}^a{}_{bcd} &= R^a{}_{bcd} - 2\nabla_{[d} C^a_{c]b} + 2C^m_{b[d} C^a_{c]m} \\ &= R^a{}_{bcd} + 2\delta^a_{[d} \nabla_{c]} \nabla_b \ln\Omega + 2g^{am} g_{b[d} \nabla_{c]} \nabla_m \ln\Omega \\ &\quad +2 \left(\nabla_{[d} \ln\Omega\right) \delta^a_{c]} \nabla_b \ln\Omega - 2\left(\nabla_{[d} \ln\Omega\right) g_{c]b} g^{am} \nabla_m \ln\Omega \\ &\quad -2g_{b[d} \delta^a_{c]} g^{mn} \left(\nabla_m \ln\Omega\right) \nabla_n \ln\Omega. \end{aligned} \tag{D.0.12}$$

Contracting on a and c, we obtain the Ricci tensor

$$\begin{aligned} \widetilde{R}_{ab} &= R_{ab} - (D-2)\, \nabla_a \nabla_b \ln\Omega - g_{ab} g^{mn} \nabla_m \nabla_n \ln\Omega \\ &\quad + (D-2)\left(\nabla_a \ln\Omega\right) \nabla_b \ln\Omega - (D-2)\, g_{ab} g^{mn} \left(\nabla_m \ln\Omega\right) \left(\nabla_n \ln\Omega\right) \end{aligned} \tag{D.0.13}$$

where D denotes the dimension of the manifolds involved. Transvection of the last equation with $\widetilde{g}^{ab} = \Omega^{-2} g^{ab}$ gives the relation between the scalar curvatures,

$$\widetilde{R} = \Omega^{-2} \left[R - 2 \left(D - 1 \right) \Box \ln \Omega - \left(D - 1 \right) \left(D - 2 \right) g^{ab} \left(\nabla_a \ln \Omega \right) \nabla_b \ln \Omega \right] \tag{D.0.14}$$

where $\Box = g^{ab} \nabla_a \nabla_b$.

This ends the necessary background on conformally related space-times.

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