

The nature of cosmological singularities in isotropic universes and braneworlds

by

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Abstract

In this thesis we explore the character of finite-time singularities that are possible to arise in FRW universes and in specific braneworld configurations.

We begin with a review of the structure and properties of RW-spaces. We then give an adaptation of a completeness theorem and analyse complete model universes from the recent literature.

We continue with the construction of a classification plan of singularities that may appear in FRW universes. Our tool for this quest is the contrapositive of the completeness theorem which translates in *necessary* conditions for singularities. The classification derived in this way is based on the behaviour of the Hubble rate and offers a first insight into the nature of singularities. We further extend and complete this classification with the use of the Bel-Robinson energy. We also associate the existence of a closed trapped surface with a divergence in the Bel-Robinson energy and use this relation to determine the evolution of various specific cosmologies.

Next we study models from the recent literature that offer the appropriate physical conditions for the appearance of singularities predicted by our classification. This study shows that our classification not only accommodates all known types of singularities but it also predicts new ones.

Finally, we focus on braneworld models consisting of a three-brane embedded in a five-dimensional bulk space that is inhabited by a scalar field or a perfect fluid. Our goal is to study the singularities that may occur within a finite distance from the brane. Our analysis that is performed with the method of asymptotic splittings shows that these braneworlds can exhibit three main types of finite-distance singularities.

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Chapter 1

Introduction

Ever since the fundamental discovery of the big bang and big crunch singularities in the general relativistic FRW universes of dust and radiation a lot of effort has been focused on finding ways to avoid these philosophically and physically puzzling features of spacetime structure. Indeed, spacetime singularities constitute a challenging issue philosophically since they imply that the immense universe emerged a finite time ago through a region of spacetime of infinite curvature and zero size, as well as physically since general relativity fails to describe such a state of spacetime. A common way out of this situation is to modify general relativity to obtain theories of gravity that do not possibly contain singularities. A different attitude is to accept that singularities are essential features of our universe and investigate further conditions that trigger their existence as well as those that describe their nature.

The singularity theorems of general relativity [1] gave appropriate conditions that led to the existence of singularities in generic spacetimes; they did not, however, provide detailed information about their nature. Uncovering the nature of spacetime singularities remains therefore an open issue in cosmology and this thesis focuses on the precise pathologies that may determine the character of singularities in the class of isotropic spacetimes. Before we analyse more carefully our goals, let us briefly review the development of recent research on the issue of finite-time singularities that partly motivates our own research reported

herein.

An investigation of the recent literature actually shows that interest in the issue of spacetime singularities has been renewed through the construction of various cosmological models with finite-time singularities. Among these are those similar to the standard big bang and big crunch singularities ones albeit appearing in universes containing different and even ‘exotic’ forms of matter, as well as other kinds not met in the standard cosmological models studied in the usual textbook expositions of cosmology.

Interestingly, these novel types of singularities sometimes involve the presence of quite an exotic type of matter (cf. [2]-[47]), a characteristic example of such a singularity being the so-called big-rip singularity (cf. [3], [14], [38], [48], [49], [50], [51], [52], [53]). The study of some of these newer types of singularities started off as soon as data from the WMAP (cf. [54], [55]) became available. These data indicate that the universe is currently undergoing an accelerating phase typically attributed to a form of *dark energy* that may be described by a linear equation of state with a parameter w converging, however, towards the value $w = -1$. The case with $w < -1$ is not ruled out by the data [3] and this provides a surprising implication for the future of the universe which had not been appreciated earlier: once such a dark energy component becomes dominant, the universe is destined to face up with a violent end towards which the scale factor, the Hubble expansion rate, the horizon distance, the density and the pressure all experiencing a divergence. In this sense every bound structure will be *ripped apart*.

The type of dark energy that leads to the emergence of the big-rip type of singularity mentioned above has some awkward characteristics (see Section 4.3) but it nonetheless sparked new interest on finite-time singularities. In particular, research focused (cf. [49], [50], [56]-[66]) in finding paradigms of cosmologies that exhibited future finite-time singularities potentially distinct from big crunches and rips but still feasible to arise in an expanding phase of a model universe which did not require exotic forms of matter. This led to the discovery of *sudden* and other milder types of singularities. These singularities are characterised by the fact that the second or higher order derivatives of the scale factor

diverge while the scale factor itself and the Hubble rate remain finite.

The models described in the preceding paragraphs played the important role of providing the appropriate physical circumstances for the appearance of such singularities. But still they provide mere *examples* not general theorems that could capture the unifying reasons for the occurrence of such spacetime singularities. In fact, the study of an increasing number of models accommodating finite-time singularities leaves one with the impression that different singular behaviours are totally determined by (or connected to) the particularities of each model.

This situation, however, provides us with a fertile ground for the formulation of some basic questions such as: What do all these different singular cosmologies have in common? Is there some underline reason that unifies all these ‘phenomenologically’ different singular behaviours? Are there other types of finite-time singularities which are possible in these models but have not yet been uncovered? What are the general criteria that predict and describe the possible kinds and nature of finite-time singularities?

This thesis aims to provide answers to such questions. We will show that singular isotropic universes are characterised by the basic property that their Hubble expansion rate *fails* in one way or another to be an integrable function of the proper time for an infinite time-interval. The various ways in which this non-integrability can be implemented leads to a classification of finite-time singularities that consists of three possible types which accommodate all the recently introduced singularities mentioned above [67], [68]. This result offers first evidence for a more detailed description of the nature of singularities in isotropic spacetimes.

Note that such a ‘classification’ of singularities is totally effected by the geometry of the spacetime in question; in no way do the various matter fields influence its construction. A more subtle examination of the character of a singularity implies however that matter fields may strongly affect the asymptotic behaviour of these model universes and hence play a role in changing the nature of their singularities. This type of classification needs therefore to be refined by somehow incorporating the various matter fields into the picture.

This refinement results in a more detailed classification scheme that sharpens our view of the nature of singularities [69]-[71].

Our approach in constructing a classification of singularities is a geometric one. In particular, for the Hubble rate classification mentioned above we make use of the contrapositive of the completeness theorem of [72], [73], while for the second we use the notion of the Bel-Robinson energy (an invariant geometric quantity representing the slice-projected energy of the gravitational field). This latter approach makes our classification conducive to more general spacetimes as well as more flexible since now more functions describe the possible different types of singularities and trace the properties and structure of spacetime.

After the completion of the classification of singularities in isotropic universes, this thesis focuses on the analysis of singularities that occur in the different context of *braneworld universes*. The study of these models (originally motivated by string theory) offers many interesting scenarios in cosmology and particle physics (see Section 5.1 for a short review). Our interest in this recently issued research area originates from the fact that these models exhibit finite-*distance* singularities that have many analogies with the finite-time singularities predicted in our classification and not met in simpler, more standard models.

For instance, the specific braneworld model of [74] captured our interest precisely because it involves the appearance of a finite-distance singularity. This model consists of a flat three-brane sitting at a fixed position of an extra spatial dimension, and a scalar field propagating throughout the whole of the enveloping five-dimensional space. At the vicinity of the singularity occurring within a finite-distance from the brane, the energy density of the scalar field becomes infinite and this may offer an explanation to the small, observed value of the vacuum energy (actualised in this model by the scalar field).

We further explore whether this singularity continues to appear when we consider a curved instead of a flat brane. Our preliminary results indicate that the singularity is in fact removed at an infinite distance from the brane [75]. This work is further extended here by examining other types of finite-distance singularities that become possible once we consider a different form of bulk matter such as a perfect fluid.

By the use of the method of asymptotic splittings developed in [76], we find that there are three possible types of finite-distance singularities, triggering an infinite leak of energy of the matter that inhabits the background space.

A more detailed plan of this thesis is as follows:

In Chapter 2, after a review of the basic properties of Robertson-Walker spaces we give an adaptation of the proof of the general completeness theorem of [72] for the specific case of such spaces and analyse complete model universes from the recent literature.

In Chapter 3, we use the contrapositive of the completeness theorem [72], [73] to derive *necessary* conditions for singularities in isotropic spacetimes. These in turn provide us with a first classification plan of singularities. This plan is later extended and completed with the use of the Bel-Robinson energy. We also associate the existence of a closed trapped surface with a divergence in the Bel-Robinson energy and use this relation to determine the evolution of various specific isotropic models.

In Chapter 4, we exploit model universes appearing in the recent literature to demonstrate how some particular members of our classification indeed arise during the evolution of such models. It follows from the analysis in this chapter that our classification scheme accommodates all known types of singularities. In fact, an examination of the literature shows that many new types of singularities -predicted by our classification- have not yet been embodied in appropriate cosmological models.

In Chapter 5, we present an asymptotic analysis of braneworld models that consist of a three-brane embedded in a five-dimensional bulk space filled with a scalar field or a perfect fluid. Our analysis, performed with the method of asymptotic splittings, shows that these models are constrained to exhibit only three main types of singularities within a finite distance from the position of the brane which is taken to be at the origin.

In Chapter 6 we conclude and discuss projects of future work. This thesis has also five appendices. In Appendix A, we explain the notation and conventions that we use throughout, and in the appendices that follow we give some background of some of the basic concepts that we use in this work such as the extrinsic curvature and the Gauss-Codazzi

formalism (Appendix B), the Bel-Robinson energy (Appendix C), the energy-momentum tensor of a scalar field (Appendix D) and the method of asymptotic splittings (Appendix E).

We declare that all statements in this work correspond to original research conducted by the author except in all those places where it is otherwise stated and proper citations are then given.

Chapter 2

Completeness and Robertson-Walker geometry

In this Chapter we explore the character of Robertson-Walker spaces, the basic spacetimes used and studied in this work. We start by reviewing, in Section 2.1, the motivation behind the adaptation of such spaces, their basic properties and structure as well as some simple features of singularities that they may exhibit. In Section 2.2, we first review the assumptions of the completeness theorem stated and proved in [72] for generic spacetimes and then give a version of that proof for the case of a Robertson-Walker space. The key idea of the proof of that theorem is to study the geodesic equations satisfied by every causal geodesic; this is a different approach from the one taken in the proof of the ‘antipodal’ singularity theorems, where the idea is to study instead the Raychaudhuri equation which describes the expansion of a bundle of geodesics. The completeness theorem applied to the case of a Robertson-Walker space guarantees that the integrability of the norm of the extrinsic curvature, or equivalently of the Hubble expansion rate, on an infinite-proper time interval suffices for the completeness of such spaces. Finally, in Section 2.3 we analyse complete universes from the recent literature.

2.1 Structure and properties of Robertson-Walker spaces

Since observations on the large-scale structure of the universe are limited, the construction of cosmological models includes at a most basic level philosophical ideas that we accept as principles. The *cosmological principle* plays a central role in the construction of model universes and can be viewed as a generalisation of the Copernican principle according to which the earth does not occupy a privileged position in the solar system. The cosmological principle assumes that in turn the solar system, our galaxy, and our local group of galaxies do not occupy a privileged position in the universe.

The cosmological principle therefore implies that the universe on a large enough scale ought to be spatially homogeneous. A spacetime is spatially homogeneous if at any instant of time, represented by a spacelike hypersurface Σ_t , any two points on Σ_t are equivalent. More precisely, a spatially homogeneous spacetime can be foliated by a one-parameter family of spacelike hypersurfaces Σ_t such that for each t and for any pair of points p, q on Σ_t , there exists an isometry that maps p into q (cf. [77], pp. 92-93). This implies that the curvature of Σ_t cannot vary from point to point i.e., the spacelike slices Σ_t are surfaces of constant curvature.

Apart from imposing homogeneity, the cosmological principle also requires isotropy about every point in spacetime; this means that there are no privileged directions, or equivalently, that the universe is spherically symmetric about every point.

The strongest evidence for isotropy is provided by the discovery of Penzias and Wilson [78] that the universe is filled with a microwave background radiation of about 3 K that is highly isotropic. Since isotropy implies homogeneity [79], this fundamental discovery provided evidence for the validity of the cosmological principle.

The requirement of isotropy further implies that there exists a class of privileged observers, the *isotropic observers*, that move along a congruence of timelike geodesics with tangent vectors orthogonal to the homogeneous slices Σ_t . These timelike geodesics do not intersect, except possibly at singular points, so that through every non-singular point in spacetime passes at *most* one isotropic observer. The orthogonality between these geodesics

and the slices Σ_t is a necessary feature of isotropic spacetimes: if these geodesics did not cut the surfaces Σ_t orthogonally, then we would be able to construct a preferred spatial direction in violation of isotropy (cf. [77], p. 93).

The suitable metric for the description of an isotropic spacetime is the Robertson-Walker metric given by

$$ds^2 = -dt^2 + a^2(t)d\Omega^2, \quad (2.1.1)$$

where $d\Omega^2$ is the metric of the three-spaces of constant negative, zero or positive curvature. The non-zero constant curvature can be normalised to be ± 1 , (so that $d\Omega^2$ is the metric of the unit three-sphere or of the unit hyperbolic three-space respectively, and the trichotomy of possibilities reads as follows:

$$ds^2 = -dt^2 + a^2(t)(dr^2 + f^2(r)(d\theta^2 + \sin^2\theta d\phi^2)), \quad (2.1.2)$$

where $a(t)$ is the *scale factor*, and

$$f(r) = \begin{cases} r, & k = 0 \\ \sin r, & k = 1 \\ \sinh r, & k = -1, \end{cases} \quad (2.1.3)$$

with $r \in [0, \infty)$ if $k = 0, -1$, whereas $r \in [0, 2\pi]$ if $k = +1$. We call a universe with constant curvature k equal to $-1, 0$, or $+1$, an open, flat, or closed universe respectively. Note that the spacelike slices of a flat or open universe are of an infinite extent in contrast to those of a closed universe which are finite.

As we have already remarked already in a isotropic universe we are always able to construct a privilege class of isotropic observers. Each of these observers can be thought of as a particle of a cosmic fluid filling spacetime; the world lines of the observers become then the flow lines of the fluid. Because of the orthogonality between these flow lines and the spacelike slices Σ_t , the spacelike coordinates of a particle in the fluid are constant *-co-moving coordinates-* along these lines. This cosmic fluid models the matter content of

the universe and the scale factor describes the separation of neighboring flow lines (cf. [1] p.136). In the simplest case the energy-momentum tensor of the cosmic fluid is thus taken to be that of a perfect fluid with density μ and pressure p which are functions only of the (proper) time. The resulting Einstein-Friedman equations can be written in the form

$$3H^2 = \mu + 3\frac{k}{a^2}, \quad (2.1.4)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\mu + 3p), \quad (2.1.5)$$

while the conservation of the energy-momentum tensor gives

$$\dot{\mu} + 3H(\mu + p) = 0, \quad (2.1.6)$$

where $H = \dot{a}/a$ is the Hubble rate.

In the absence of exotic forms of matter, we assume that $\mu > 0$ and $p \geq 0$ for all t . Then Eq. (2.1.5) shows that \dot{a} cannot be constant and thus the universe is either expanding or contracting. The discovery of the expansion of the universe by Hubble in 1929 offered a very interesting implication for its history: because its density decreases as the universe expands (see Eq. (2.1.6)), it follows that it should have been infinite in the past as $a(t) \rightarrow 0$; the universe must have therefore began its existence with a singularity. In fact, the singularity theorems predict that this is inevitable in all models satisfying $\mu + 3p > 0$.

The first cosmological models with a RW geometry that were found to possess such a singularity were the dust and radiation-filled universes. Dust is a pressureless fluid that provides a good approximation for a matter component comprised solely by non-interacting galaxies. This follows because the observed relative velocities of galaxies in groups are small and hence galaxies are interpreted as particles of a pressureless fluid. On the other hand, an adequate approximation to the matter content of the universe in an early epoch is provided by radiation which is described by taking $p = \mu/3$. Away from an initial singularity open and flat universes typically expand forever, whereas, closed ones end their existence with

another singularity that arises within a finite time from the initial big bang.

2.2 Completeness theorems

Although we are going to focus on exclusively isotropic spacetimes, we refer briefly in this Section to the basic properties of general spacetimes used in the formulation of the completeness theorem of [72]. In the end of this Section, we give an adapted version of the original theorem and proof of [72], for our case of interest -isotropic spacetimes. We will refer to the original version of the completeness theorem (Theorem 2.2.1 below) later in Chapter 3 when we shall discuss the possible extensions of our work to more general spacetimes.

We begin by giving some basic definitions. Consider a spacetime (\mathcal{V}, g) with $\mathcal{V} = \mathcal{M} \times \mathcal{I}$, $\mathcal{I} = (t_0, \infty)$, where \mathcal{M} is a smooth manifold of dimension n and ${}^{(n+1)}g$ a Lorentzian metric which in the usual $n + 1$ splitting, reads

$${}^{(n+1)}g \equiv -N^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \quad (2.2.1)$$

Here $N = N(t, x^i)$ is called the *lapse function*, $\beta^i(t, x^j)$ is called the *shift function* and the spatial slices $\mathcal{M}_t (= \mathcal{M} \times \{t\})$ are spacelike submanifolds endowed with the time-dependent spatial metric $g_t \equiv g_{ij} dx^i dx^j$. We call such a spacetime a *sliced space* [80]. A sliced space is time-oriented by increasing t and we choose $\mathcal{I} = (t_0, \infty)$ (or $\mathcal{I} = (-\infty, t_0)$) when we study the future (past) singularity behaviour of an expanding (contracting) universe with a singularity in the past (future), for instance at $t = 0 < t_0$ ($t = 0 > t_0$). However, since t is just a coordinate, our study could apply as well to any interval $\mathcal{I} \subset \mathbb{R}$.

A natural causal assumption for (\mathcal{V}, g) is that it is *globally hyperbolic*. This implies the existence of a time function on (\mathcal{V}, g) . In a globally hyperbolic space, spacetime splits as above with each spacelike slice \mathcal{M}_t being a Cauchy surface; this means that each timelike and null curve without end points cuts \mathcal{M}_t exactly once [81]. In such a spacetime therefore the future and history of the universe can be predicted or retrodicted from conditions at

the instant of time represented by \mathcal{M}_t (cf. [77], pp.200-202).

Following the definitions given in [72], we say that a sliced space has *uniformly bounded lapse* if the lapse function N is bounded below and above by positive numbers N_m and N_M ,

$$0 < N_m \leq N \leq N_M. \quad (2.2.2)$$

A sliced space has *uniformly bounded shift* if the g_t ($t \in \mathcal{I}$) norm of the shift vector β , projection on the tangent space to \mathcal{M}_t of the tangent to the lines $\{x\} \times \mathcal{I}$, is uniformly bounded by a number B .

A sliced space has *uniformly bounded spatial metric* if the time-dependent metric $g_t \equiv g_{ij}dx^i dx^j$ is uniformly bounded below for all $t \in \mathcal{I}$ by a metric $\gamma = g_{t_0}$, that is there exists a number $A > 0$ such that for all tangent vectors v to \mathcal{M} it holds that

$$A\gamma_{ij}v^i v^j \leq g_{ij}v^i v^j. \quad (2.2.3)$$

Definition 2.2.1 *A sliced space (\mathcal{V}, g) with uniformly bounded lapse, shift and spatial metric is called regularly sliced.*

Denoting by ∇N the space gradient of the lapse N , by K_{ij} the extrinsic curvature of \mathcal{M}_t , and by $|K|_g^2$ the product $g^{am}g^{bn}K_{ab}K_{mn}$, we have the following theorem of [72] which gives sufficient conditions for geodesic completeness:

Theorem 2.2.1 *Let (\mathcal{V}, g) be a sliced space such that the following assumptions hold:*

C1 *(\mathcal{V}, g) is globally hyperbolic*

C2 *(\mathcal{V}, g) is regularly sliced*

C3 *For each finite t_1 , the space gradient of the lapse, $|\nabla N|_g$, is bounded by a function of t which is integrable on $[t_1, +\infty)$*

C4 *For each finite t_1 , $|K|_g$ is bounded by a function of t which is integrable on $[t_1, +\infty)$.*

Then (\mathcal{V}, g) is future timelike and null geodesically complete.

It was later proved in [80] that in a regularly sliced spacetime, condition C1 is in fact *equivalent* to the condition that each slice of (\mathcal{V}, g) is a complete Riemannian manifold.

Turning back now to isotropic spacetimes we see from the definition of a sliced space and the form of the RW metric (Eqs. (2.1.2), (2.1.3)), that an isotropic spacetime is a sliced space with $N = 1$, $\beta = 0$. Following the Definition (2.2.1) we note that:

Remark 2.2.1 *An isotropic spacetime is regularly sliced iff the scale factor is a bounded from below function of time.*

We can now reformulate the completeness theorem 2.2.1 so as to apply directly to the case of an isotropic spacetime.

Theorem 2.2.2 (Completeness of RW spacetimes) *Every globally hyperbolic, regularly sliced RW space such that for each finite t_1 the norm of the extrinsic curvature is integrable on $[t_1, +\infty)$, is future timelike and null geodesically complete.*

PROOF.

Consider a causal geodesic in an isotropic universe. The tangent vector to these geodesics $u^a = dx^a/ds$, where $x^a = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ and s is the proper time or an affine parameter measured along a timelike or null geodesic respectively, satisfies the geodesic equations

$$\frac{d^2 x^a}{ds^2} + \Gamma_{mn}^a \frac{dx^m}{ds} \frac{dx^n}{ds} = 0. \quad (2.2.4)$$

The $a = 0$ component of these equations for this type of universe is

$$\frac{d^2 t}{ds^2} + \Gamma_{ij}^0 \frac{dx^i}{ds} \frac{dx^j}{ds} = 0, \quad (2.2.5)$$

where $i, j = 1, 2, 3$ (the rest of the Christoffel symbols Γ_{0i}^0 and Γ_{00}^0 vanish).

Dividing Eq. (2.2.5) by $(dt/ds)^2$ we find,

$$\frac{ds}{dt} \frac{d}{dt} \left(\frac{dt}{ds} \right) + \Gamma_{ij}^0 \frac{dx^i}{dt} \frac{dx^j}{dt} = 0. \quad (2.2.6)$$

Setting $y = dt/ds$ and denoting by \dot{y} the derivative of y with respect to t , the above equation is written in the form

$$\frac{\dot{y}}{y} = -\Gamma_{ij}^0 \frac{dx^i}{dt} \frac{dx^j}{dt}. \quad (2.2.7)$$

Integrating over the interval $[t_1, t]$, we find

$$\ln \frac{y(t)}{y(t_1)} = - \int_{t_1}^t \Gamma_{ij}^0 v^i v^j dt, \quad (2.2.8)$$

where we have set $v^i = dx^i/dt$. Since $\Gamma_{ij}^0 = K_{ij}$ ¹ we can write Eq. (2.2.8) as follows:

$$\ln \frac{y(t)}{y(t_1)} = - \int_{t_1}^t K_{ij}^0 v^i v^j dt. \quad (2.2.9)$$

Substitution then in Eq. (3.16) of [72] gives

$$\ln \frac{y(t)}{y(t_1)} < \int_{t_1}^t |K|_g dt = \sqrt{3} \int_{t_1}^t \left| \frac{\dot{a}}{a} \right| dt, \quad (2.2.10)$$

where in the last equation we have used Eq. (B.1.15) from Appendix B. Using the hypothesis of the integrability of $|K|_g$, we see that $y = dt/ds$ is uniformly bounded. This means that ds/dt is bounded away from zero and hence the length of the geodesic given by²

$$\int_{t_1}^{\infty} \frac{ds}{dt} dt \quad (2.2.11)$$

is infinite. This implies that the spacetime is causally geodesically complete. ■

Past completeness is derived by exchanging future with past and $[t_1, +\infty)$ with $(-\infty, t_1]$. It follows from the above theorem that a globally hyperbolic and regularly sliced isotropic spacetime is complete provided that the spacelike slices Σ_t do not curve too much as they are seen by an observer located outside Σ_t in spacetime, for all $t \in [t_1, +\infty)$.

Now since the norm of the extrinsic curvature in an isotropic spacetime is proportional

¹Using Eqs. (B.1.14)-(B.1.14) we can verify that $\Gamma_{11}^0 = \dot{a}a = K_{11}$, $\Gamma_{22}^0 = \dot{a}af^2 = K_{22}$, $\Gamma_{33}^0 = \dot{a}af^2 \sin^2 \theta = K_{33}$.

²As it was shown in [72] a future-directed causal curve $C : s \mapsto C(s)$ can be reparametrised by t : if $n = (-1, 0)$ is the timelike normal to \mathcal{M}_t then $g\left(\frac{dC}{ds}, n\right) = -\frac{dt}{ds} < 0$ and hence $\frac{dt}{ds} > 0$.

to the Hubble expansion rate $H(t)$ (see Appendix B, Eq. (B.1.15)), we conclude that such a spacetime is complete if it exhibits an expansion rate that is an integrable function of proper time on $[t_1, +\infty)$.

In the following Section, we will see that there exist complete FRW models that have an expansion rate which is non-integrable on $[t_1, +\infty)$. This implies that the integrability of the Hubble rate on an infinite proper-time interval is a sufficient but *not* a necessary condition for completeness. In order to prove completeness of these universes we take a different approach in which we make use of the Bel-Robinson energy.

As we discuss in Appendix C, the Bel-Robinson energy is a kind of energy of the gravitational field *projected* in a sense to a slice in spacetime. In particular, for a RW metric the Bel-Robinson energy at time t , $\mathcal{B}(t)$, is given by

$$\mathcal{B}(t) = \frac{C}{2} (|E|^2 + |D|^2), \quad (2.2.12)$$

where C is the constant volume *of* (or *in*, in the case of a non-compact space) the three dimensional slice at time t and $|E|$, $|D|$ are the norms of the two electric tensors, namely,

$$|E|^2 = 3 \frac{\ddot{a}^2}{a^2}, \quad (2.2.13)$$

$$|D|^2 = 3 \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right)^2. \quad (2.2.14)$$

It can be proved that a closed or flat, expanding at t_* , FRW universe that has a bounded Bel-Robinson energy is causally geodesically complete [82]. Therefore complete FRW models which fail to have a Hubble expansion rate that is integrable on an infinite proper-time interval may still manage to be complete provided that they have a bounded Bel-Robinson energy. In the following Section we will illustrate this result with the analysis of various models.

2.3 Complete FRW cosmologies

Completeness is not a property only shared by ‘normal’ cosmologies. Although later in Section 4.3, we meet isotropic universes filled with dark energy that end their existence in a big-rip singularity, this need not always be the case. There are many examples of such ‘exotic’ universes containing dark energy that still manage to evade the big-rip singularity and exist forever. This is achieved either by assuming that the dark energy component satisfies simultaneously two different equations of state, or that it satisfies a generalised equation of state that includes matter ranging from Chaplygin gasses to perfect fluids. We shall presently analyse examples of complete universes belonging to both of these categories in turn.

For the first case, consider the flat FRW universe studied in [83] (see also [84]). In this case the dark energy component satisfies a ‘phantom’ equation of state of the form

$$p = w\mu, \quad \text{with } w = w(t) \quad \text{and} \quad w < -1,$$

as well as an equation of state of a *Chaplygin gas* (cf. [85], [86])

$$p = -\frac{A}{\mu}, \quad \text{where } A > 0, \quad \text{constant},$$

(similar results will hold when $w \in (-1, -1/3)$ *k*-essence models, see, e.g., [84]). Substituting only the latter equation of state in the continuity equation, Eq. (2.1.6), and integrating we find

$$\mu^2(t) = A + (\mu_0^2 - A) \left(\frac{a_0}{a(t)} \right)^6. \quad (2.3.1)$$

We can now take into account the phantom equation of state and reform the above equation. This is done by calculating, with the use of both equations of state, the time-dependent parameter $w(t)$,

$$w(t) = -\frac{A}{\mu^2}. \quad (2.3.2)$$

Substituting this back in Eq. (2.3.1), we find the following form for the energy density,

$$\mu(t) = \mu_0^2[-w_0 + (1 + w_0)(a_0/a(t))^6]^{1/2}, \quad (2.3.3)$$

where $A = -w_0\mu_0^2$ and $w_0 < -1$, which we can now insert in the Friedmann equation, Eq. (2.1.4), and derive the following solution for the scale factor:

$$a(t) = (Ce^{C_1(t-t_0)})^{1/6}, \quad (2.3.4)$$

where C and C_1 are positive constants. This model admits cosmic acceleration, $\ddot{a} > 0$, and $a(t) \rightarrow \infty$ as $t \rightarrow \infty$ and it is geodesically complete. The Hubble parameter takes the constant value,

$$H = \frac{C_1 C}{6}. \quad (2.3.5)$$

Here also the Bel-Robinson energy remains always finite and hence by the completeness theorem in [82] we conclude that the model is geodesically complete.

As we mentioned in the beginning of this section, the second technique to trigger the avoidance of a big-rip singularity in models containing dark energy is to assume that the dark energy satisfies a *generalised* Chaplygin equation of state. This was done in [84] for a flat universe. In particular, the equation of state is here chosen so that

$$p = -\mu^{-\alpha}[C + (\mu^{1+\alpha} - C)^{\alpha/(1+\alpha)}], \quad (2.3.6)$$

where $C = A/(1 + w) - 1$, $1 + \alpha = 1/(1 + w)$, and α is a real parameter. Note that in the limit $w \rightarrow 0$ the gas reduces to a Chaplygin gas satisfying $p = -A\mu^{-\alpha}$, whereas as $A \rightarrow 0$ it takes the form of a perfect fluid with equation of state $p = w\mu$. The scale factor is given by the form

$$a(t) = (C_1 e^{-C_3(t-t_0)} + C_2 e^{C_3(t-t_0)})^{2/3}, \quad (2.3.7)$$

where C_1 , C_2 and C_3 are parameters depending on a_0 (the initial value of the scale factor), A , w and B , with B being a positive integration constant. At the asymptotic limits $t \rightarrow t_0$ and $t \rightarrow \infty$, we find that H tends to suitable constants, that is it remains finite on $[t_0, \infty)$.

The complete character of this model is guaranteed by a clearly bounded Bel-Robinson energy.

Assuming the existence of more conventional matter contents such as radiation and a cosmological constant in a closed FRW universe, we obtain the following solution originally given in [87],

$$a(t) = a_i \left[1 + \exp(\sqrt{2}t/a_i) \right]^{1/2}, \quad (2.3.8)$$

where a_i is a constant that bounds $a(t)$ from below making this universe regularly hyperbolic. One then finds that H becomes zero as $t \rightarrow -\infty$, it is integrable on $(-\infty, 0]$ and the Bel-Robinson energy is bounded. Thus we conclude that this universe is geodesically complete, either using the result of [82] or alternatively by the completeness theorem 2.2.2.

Chapter 3

Classification of singularities

3.1 Introduction

Up to now we studied situations where geodesic completeness plays a major role. From now on we shall focus on the other important issue, namely, that of gravitational singularities. Spacetime singularities characterise solutions describing cosmological and gravitational collapse situations in general relativity and in other metric theories of gravity. Understanding the various aspects of spacetime singularities constitutes the so-called singularity problem. Viewed from a mathematical perspective the singularity problem apart from being appealing because of its fundamental nature, it is also a pressing one, ever-present in manifolds and metrics of arbitrary form under general topological and geometric assumptions (see Section 3.4). From a physical perspective the singularity problem attracts attention because of the wide variety of cosmological models in general relativity and other metric theories of gravity that exhibit singularities that need to be analysed and understood (see Sections 4.2-4.5 for a detailed analysis of singular cosmological models in general relativity).

Even though spacetime singularities represent an extensively studied area from both a mathematical as well as a physical point of view, there are still many important unresolved issues in this general area. It is essential to find appropriate tools that will serve to define and analyse the structure and nature of spacetime singularities. This will hopefully lead to

a more basic understanding of all those qualities that make up the character of singularities along with a more detailed classification of them. But there exists a risk in performing such a project: mapping out only a part of the field comprised of all possible types of singularities would provide us with misleading conclusions about the generality of the types found since we would then be shedding light only on some aspects of the field while others would remain concealed [88]. It is thus important to attempt to plot a more or less *complete* classification scheme. Any such scheme ought to acknowledge both the mathematical and the physical qualities of a spacetime singularity. Among the mathematical qualities the most important one is that of geodesic completeness/incompleteness and it should therefore play a key role in the planning of a classification. On the other hand, physical qualities describe the way matter fields behave on approach to the singularity and are expected to contribute significantly to our understanding of its nature.

We begin the analysis of singularities in this chapter by focusing on those types of singularity that may arise in the simplest class of isotropic spacetimes. The tools that we use are of a geometric nature. Our geometric approach on one hand makes the resulting classification scheme easily adjustable to a further future study of more complicated spacetimes, and on the other it helps us find complementary results and generic properties of the class of spacetimes being analysed.

Our attack to a realisation of a complete classification scheme as discussed above is implemented in two stages. In the *first stage*, we exploit the contrapositive of the completeness theorem Theorem 2.2.1 in the effort to derive *necessary* conditions for the existence of singularities in the case of isotropic cosmological models. This provides us with a classification of singularities that is solely based on the behaviour of the Hubble expansion rate H .

For various reasons to be given later, the classification depicted in the first stage does not provide us with all details necessary for a complete classification scheme referred to above. We therefore need a complementary tool in order to bring to surface all the other missing factors and complete our plan.

This leads us to the *second stage* which consists of a classification scheme based on the introduction of a new invariant, the Bel-Robinson energy, a geometric quantity built out of combinations of various curvature components that traces successfully all those required qualities missing from the first classification.

Apart from being involved in the second stage of our classification scheme, we show that the use of the Bel-Robinson energy in combination with the notion of a closed trapped surface induces important information about the dynamical evolution of isotropic models. In particular, we prove that the existence of a closed trapped surface can be traced in the dynamical behaviour of the Bel-Robinson energy. We show by analysing various examples from the literature that when the existence of a closed trapped surface leads to a singularity the Bel-Robinson energy also diverges, whereas in the opposite case the Bel-Robinson energy remains finite and signatures the complete character of the model.

The plan of this chapter is as follows: In Section 3.2, we classify singularities according to the behaviour of the Hubble rate. In Section 3.3 we analyse the necessity for refining our classification scheme: In Subsection 3.3.1 we derive a complete classification scheme based on the possible behaviours of three functions the Hubble rate, the scale factor and the Bel-Robinson energy. In Section 3.4 we analyse the interplay of closed trapped surfaces and the dynamical behaviour of the Bel-Robinson energy: In the Subsection 3.4.1 we show how the existence of a closed trapped surface is linked to a possible divergence in the Bel-Robinson energy and in Subsection 3.4.2 we illustrate this result for various model universes. Finally, in Section 3.5 we discuss our conclusions.

3.2 Hubble parameter singularities

Theorem 2.2.2 provides us with sufficient conditions for completeness and we can therefore use its contrapositive to derive necessary conditions for the existence of singularities. This will also give us an insight into the character of possible singularities. In particular, consider a singular, globally hyperbolic and regularly sliced isotropic universe. Then according to Theorem 2.2.2, there is a finite time t_1 for which H fails to be integrable on the proper time

interval $[t_1, \infty)$. In turn, this non-integrability of the expansion rate H can be implemented in different ways and we arrive at the following result which provides first evidence for the nature of finite-time singularities, based entirely on the behaviour of the Hubble parameter [67], [68].

Theorem 3.2.1 *Necessary conditions for the existence of finite time singularities in globally hyperbolic, regularly sliced isotropic universes are:*

S₁ *For each finite t , H is non-integrable on $[t_1, t]$, or*

S₂ *H blows up in a finite future time, or*

S₃ *H is defined and integrable for only a finite proper time interval.*

Condition S_1 describes a big bang type of singularity when H blows up at t_1 since then it is not integrable on any interval of the form $[t_1, t]$, $t > t_1$ (the condition of regular slicing is violated in this case but the scale factor is bounded from above). However, under S_1 we can have other types of singularities: Since $H(\tau)$ is integrable on an interval $[t_1, t]$, if $H(\tau)$ is defined on $[t_1, t]$, continuous on (t_1, t) and the limits $\lim_{\tau \rightarrow t_1^+} H(\tau)$ and $\lim_{\tau \rightarrow t^-} H(\tau)$ exist, the violation of *any* of these conditions leads to a singularity that is not of the big bang type discussed previously.

Condition S_2 describes a future singularity at t_s ($t_s > t_1$) characterised by the divergence of H at t_s . Condition S_3 may lead to a future singularity at t_s with H being finite on $[t_a, t_s]$ where $t_a \geq t_1$ but for this to be a genuine type of singularity, in the sense of geodesic incompleteness, one needs to demonstrate that the metric is non-extendible to a larger interval.

Note that these three conditions are not overlapping: clearly S_1 and S_2 are not overlapping with S_3 but also S_1 is not overlapping with S_2 . Indeed S_1 is not implied by S_2 for if H blows up at some finite time t_s after t_1 , then it may still be integrable on $[t_1, t]$, $t_1 < t < t_s$.

There are many examples of singularities met in the literature that belong to the types predicted by the above theorem; they are analyzed in Chapter 4 in detail. They

usually describe flat isotropic universes with various components of matter. For example, the phantom dark energy models of [51] exhibit a future singularity which falls in the S_2 category. The standard big bang singularities as well as the sudden singularity of [56] fall in the S_1 category. Other sudden singularities (which do not have a blow up of H at $t = 0$ (cf. [52])) and the inflation model of [89] both fall in the S_3 category (for a detailed analysis of these models see Chapter 4).

3.3 The (S, N) pairs

Although the classification based on the Hubble parameter is a first broad step to clearly distinguish between the various types of singularities that can occur in isotropic universes, it does not bring out some of the essential features of the dynamics that may differ from singularity to singularity. To see this we note that S_3 encloses altogether dissimilar types of singularities each characterised by a distinct pathology, for example in higher order derivatives of H , and as we have already mentioned we also accommodate distinct singularity types under S_1 . In addition, condition S_2 includes both a collapse singularity, where the scale factor $a \rightarrow 0$ as $t \rightarrow t_s$, and a singularity with $a \rightarrow \infty$ as $t \rightarrow t_s$. This type of degeneration is unwanted since it impedes the construction of a complete classification of singularities. It is therefore necessary to extend and refine this classification.

We begin by including in our classification the behavior of the scale factor. That it is only necessary to include this behaviour is seen most clearly by noticing that only in this way one may consistently distinguish between initial and final singularities, or between past and future ones, as it has been repeatedly emphasised by Penrose in his Weyl curvature hypothesis (see for example [90] pp. 765-769). According to this physical conjecture, the Weyl curvature which acts as a distorting source for matter is constrained to be almost zero at initial singularities. In contrast, at final singularities it diverges to infinity.

Taking into account the behaviour of the scale factor in our singularity classification scheme results in describing the various singularity types by a pair (S_i, N_j) , where the first component reflects the behaviour of the Hubble parameter according to Theorem 3.2.1,

while the second component symbolises one of the following three possible behaviours of the scale factor on approach to the finite-time singularity at t_s namely,

N₁ $a \rightarrow 0$ -collapse singularity or

N₂ $a \rightarrow a_s \neq 0$ -sudden singularity or

N₃ $a \rightarrow \infty$ -rip singularity.

We thus end up with nine possible singularity types given by the pairs (S_i, N_j) where $i, j = 1, 2, 3$. We further note that in the (S_2, N_3) category we can accommodate singularities having a Hubble rate which either diverges more rapidly or more slowly than the scale factor, as well as singularities with these two functions diverging almost or exactly at the same rate. Making such a further or more refined distinction between singularities that otherwise belong to the same (S, N) type means that we allow for the relative behaviours of the Hubble rate and the scale factor to emerge as we approach the singularity. This refinement can be accomplished by introducing in our analysis the following notion of *asymptotic strength*.

First recall that for the relative asymptotic behaviour of two functions f, g on approach to a point t_* , we say that:

1. $f(t)$ is *much smaller* than $g(t)$, $f(t) \ll g(t)$, if and only if $\lim_{t \rightarrow t_*} f(t)/g(t) = 0$
2. $f(t)$ is *similar* to $g(t)$, $f(t) \sim g(t)$, if and only if $0 \neq \lim_{t \rightarrow t_*} f(t)/g(t) < \infty$
3. $f(t)$ is *asymptotic* to $g(t)$, $f(t) \leftrightarrow g(t)$, if and only if $\lim_{t \rightarrow t_*} f(t)/g(t) = 1$,

There are three specific (S, N) pairs that can be further analysed according to the possible relative asymptotic behaviours of H and a :

We note that while (S_1, N_3) and (S_2, N_3) singularities can accommodate all possible relative behaviours between the Hubble rate and the scale factor, an (S_3, N_2) singularity allows only for an asymptotic or similar relative behaviour between them.

We have so far managed to derive a new classification that consists of nine types of singularities described by the pairs (S_i, N_j) where the first component reflects the behaviour of the Hubble rate (according to Theorem 3.2.1), while the second one traces the behaviour of the scale factor. This new classification can also be further refined with the use of the notion of asymptotic strength.

3.4 Bel-Robinson energy and singularities

Even though we have at this point taken care of the two main functions that describe a singularity, we still have to face another aspect of the problem of classifying singularities. We have not yet taken into account the relative behaviour of the various matter components as we approach the time singularities. A clear picture of the asymptotic behaviour of the matter fields will provide us with the physical qualities of the singularity contributing significantly to the understanding of its nature. In order to describe the behaviour of the various matter components and include it in our geometric approach, we now exploit the use of the Bel-Robinson energy by combining the behaviour of the electric parts of the Bel-Robinson energy with that of the Hubble parameter in the following way: Suppose for example that S_2 holds. Then from Eq. (C.1.10) we see that for a flat or closed universe $|D|$ necessarily becomes divergent whereas for an open universe it can be finite since the term k/a^2 may, in the case of some collapse singularities, exactly counter-balance the divergence of the H^2 term. These different cases can in turn be combined with a finite or an infinite $|E|$. Similarly, we can deduce the possible behaviours of $|E|$ and $|D|$ for a singular universe satisfying S_1 or S_3 . Thus the necessary conditions for singularities based on the behaviour of the Hubble parameter given by Theorem 3.2.1 can now be rephrased in terms of the electric parts of Bel-Robinson energy, to read:

Singularity type	Relative asymptotic strength
(S_1, N_3)	$H \gg a, H \ll a, H \leftrightarrow a, H \sim a$
(S_2, N_3)	$H \gg a, H \ll a, H \leftrightarrow a, H \sim a$
(S_3, N_2)	$H \leftrightarrow a, H \sim a$

Theorem 3.4.1 *Necessary conditions for null and timelike geodesically incomplete globally hyperbolic and regularly sliced isotropic universes are that at a finite time:*

$$\mathbf{B}_1 \quad |E| \rightarrow \infty \text{ and } |D| \rightarrow \infty$$

$$\mathbf{B}_2 \quad |E| < \infty \text{ and } |D| \rightarrow \infty$$

$$\mathbf{B}_3 \quad |E| \rightarrow \infty \text{ and } |D| < \infty$$

$$\mathbf{B}_4 \quad |E| < \infty \text{ and } |D| < \infty$$

We can now move on to list all possible types of singularities that are formed in an isotropic universe during its cosmic evolution and enumerate the possible types that result from the different combinations of the three main functions in the problem, namely, the scale factor a , the Hubble expansion rate H and the Bel-Robinson energy \mathcal{B} . These types will by necessity entail a possible blow up in the functions $|E|$, $|D|$.

If we suppose that a given model has a finite time singularity at $t = t_s$, then the possible behaviours of the functions in the triplet $(H, a, (|E|, |D|))$ in accordance with Theorems 3.2.1, 3.4.1 are as follows:

$$\mathbf{S}_1 \quad H \text{ non-integrable on } [t_1, t] \text{ for every } t > t_1$$

$$\mathbf{S}_2 \quad H \rightarrow \infty \text{ at } t_s > t_1$$

$$\mathbf{S}_3 \quad H \text{ otherwise pathological}$$

$$\mathbf{N}_1 \quad a \rightarrow 0$$

$$\mathbf{N}_2 \quad a \rightarrow a_s \neq 0$$

$$\mathbf{N}_3 \quad a \rightarrow \infty$$

$$\mathbf{B}_1 \quad |E| \rightarrow \infty, |D| \rightarrow \infty$$

$$\mathbf{B}_2 \quad |E| < \infty, |D| \rightarrow \infty$$

$$\mathbf{B}_3 \quad |E| \rightarrow \infty, |D| < \infty$$

$$\mathbf{B}_4 \quad |E| < \infty, |D| < \infty.$$

The nature of any prescribed singularity is thus described completely by specifying the components in a triplet of the form

$$(S_i, N_j, B_l),$$

with the indices i, j, l taking their respective values as above.

Note that there are a few types that cannot occur. For instance, we cannot have an (S_2, N_2, B_3) singularity because that would imply having $a < \infty$ (N_2) and $H \rightarrow \infty$ (S_2), while $3((\dot{a}/a)^2 + k/a^2)^2 < \infty$ (B_3), at t_s which is impossible since $|D|^2 \rightarrow \infty$ at t_s (k arbitrary).

A complete list of the impossible singularities is model-dependent and is generally given by triplets (S_i, N_j, B_l) where the indices in the case of a flat or a closed universe take the values $i = 1, 2, j = 1, 2, 3, l = 3, 4$, whereas in the case of an open universe they take the values $i = 1, 2, j = 2, 3, l = 3, 4$ (here by S_1 we denote for simplicity only the big bang case in the S_1 category). We thus see that some singularities which are impossible for a flat or a closed universe become possible for an open universe. Consider for example the triplet (S_2, N_1, B_3) which means having $H \rightarrow \infty, a \rightarrow 0$ and

$$|D|^2 = 3((\dot{a}/a)^2 + k/a^2)^2 < \infty$$

at t_s . This behaviour is valid only for some cases of an open universe. With these exceptions, all other types of finite time singularities can in principle be formed during the evolution of isotropic matter-filled models in general relativity or other metric theories of gravity.

This classification allows us to speak consistently of strong, mild or weak singularities. Note that the standard dust or radiation-filled big bang singularities fall under the *strongest*

singularity type, namely, the type (S_1, N_1, B_1) , since for example, in a flat universe filled with dust we have

$$a(t) \propto t^{2/3} \rightarrow 0, \quad (N_1), \quad (3.4.1)$$

$$H \propto t^{-1} \rightarrow \infty, \quad (S_1), \quad (3.4.2)$$

$$|E|^2 = 3/4H^4 \rightarrow \infty, \quad |D|^2 = 3H^4 \rightarrow \infty, \quad (B_1). \quad (3.4.3)$$

Our classification scheme is organized in such a way that the character of the singularities (i.e., the behaviour of the defining functions) becomes milder as the indices of S , N and B increase. Milder singularities in isotropic universes are thus expected to occur as one proceeds down the singularity list.

It is the purpose of this classification to apply both to vacuum as well as matter dominated models. In fact, the Bel-Robinson energy takes care in a neat way the matter case. For instance, in fluid-filled models the various behaviours B_i , $i = 1, 2, 3, 4$, of the Bel-Robinson energy density can be instead given by asymptotic conditions imposed on the density and pressure of the cosmological fluid:

$$B_1 \Leftrightarrow \mu \rightarrow \infty \text{ and } |\mu + 3p| \rightarrow \infty$$

$$B_2 \Leftrightarrow \mu \rightarrow \infty \text{ and } |\mu + 3p| < \infty$$

$$B_3 \Leftrightarrow \mu < \infty \text{ and } |\mu + 3p| \rightarrow \infty \Leftrightarrow \mu < \infty \text{ and } |p| \rightarrow \infty$$

$$B_4 \Leftrightarrow \mu < \infty \text{ and } |\mu + 3p| < \infty \Leftrightarrow \mu < \infty \text{ and } |p| < \infty.$$

We can translate these conditions to asymptotic behaviours in terms of a, H , depending on the value of k , for example,

1. If $k = 0$, $\mu < \infty \Rightarrow H^2 < \infty$, a arbitrary
2. If $k = 1$, $\mu < \infty \Rightarrow H^2 < \infty$ and $a \neq 0$
3. If $k = -1$, $\mu < \infty \Rightarrow H^2 - 1/a^2 < \infty$.

It is useful at this point to name five particular members of our classification of singularities that are of special interest.

Definition 3.4.1 *Let (S_i, N_j, B_k) be a triplet describing the type of a singularity.*

We call the triplet:

- i) (S_1, N_1, B_1) ((S_2, N_1, B_1)) a past-collapse (future-collapse) singularity.*
- ii) (S_2, N_3, B_1) a big-rip singularity.*
- ii) (S_3, N_2, B_3) a sudden singularity.*
- iv) (S_3, N_2, B_1) a sudden-rip singularity.*

These types of singularities arise in some of the most popular cosmologies appearing in the recent literature (a detailed analysis of such cosmologies is given in Chapter 4).

In order to gain a deeper insight of the dynamics on approach to the singularity we can make use of the notion of asymptotic strength defined in the previous Section. The use of asymptotic strength apart from offering us a clear view of the way that the three main functions, designating the nature of a singularity, compete asymptotically, and helping us decide about how strong one singularity type is compared to another, it also provides us with a refinement since it distinguishes singularities that fall in the same type. For example, as we have mentioned already all standard big-bang type singularities met in the literature belong to our past-collapse type (S_1, N_1, B_1) . Finding and comparing the asymptotic strengths of the singularities that may be present in dust and radiation filled universes we find that their precise nature is different since the radiation ones are characterised by the asymptotic strength $a \ll H \ll (|E| \leftrightarrow |D|)$, whereas the dust ones have $a \ll H \ll (|E| \sim |D|)$. We shall give many examples of this later in Chapter 4.

3.5 Closed trapped surfaces and Bel-Robinson energy

In this section, we focus on the possible implications of the existence of a closed trapped surface in an isotropic universe and show how its existence can be traced in the behaviour of

the Bel-Robinson energy. We also illustrate through various examples how this association can be used to shed some light on certain aspects of the dynamical evolution of isotropic models.

A closed trapped surface is a two dimensional spherical surface S in spacetime with the characteristic property that the expansion θ of both the incoming and outgoing bundle of null geodesics orthogonal to S is negative as evaluated on S . That is, the light rays that are emitted perpendicularly from and towards S converge. This is feasible when the surface S encloses enough matter to attract all the incoming and outgoing rays of light. Since nothing can travel faster than light, the matter inside S becomes trapped within the boundary of this surface. If moreover the energy conditions hold (gravity remains attractive), the radius of S will decrease gradually until a singularity is formed [1] p. 2-3.

In particular, sufficient conditions for the formation of a singularity are given by the well-known singularity theorems and include the existence of a closed trapped surface as well as appropriate energy and causality conditions imposed on the geometrical and topological structure of the spacetime (cf. [1], p. 263-267):

Theorem 3.5.1 *Spacetime (\mathcal{V}, g) cannot be null geodesically complete if:*

- (1) $R_{ab}K^aK^b \geq 0$ for all null vectors K^a
- (2) there is a non-compact Cauchy surface in \mathcal{V}
- (3) there is a closed trapped surface in \mathcal{V} .

Theorem 3.5.2 *Spacetime (\mathcal{V}, g) is not timelike and null geodesically complete if:*

- (1) $R_{ab}K^aK^b \geq 0$ for every non-spacelike vector K^a
- (2) every non-spacelike geodesic contains a point at which $K_{[a}R_{b]cd[e}K_{f]}K^cK^d \neq 0$, where K^a is the tangent vector to the geodesic
- (3) the chronology condition holds on \mathcal{V} (i.e. there are no closed timelike curves)
- (4) there exists a closed trapped surface.

We see that the formation of a closed trapped surface is triggered by the presence of a strong gravitational field. A situation where this is possible is during the gravitational collapse of a star. In particular, when a star of mass twice that of the sun has burned out all the nuclear fuel which was supporting it against the influence of its own gravity and starts collapsing, a singularity becomes inevitable after a closed trapped surface is formed. A closed trapped surface is believed to have occurred even in the past of the universe itself. Looking back in time the universe enclosed enough matter to provoke a past trapped surface that led to a past-collapse singularity in the beginning of the universe.

3.5.1 Existence

Consider a past-directed bundle of radial null geodesics in an isotropic universe with metric given by Eqs. (2.1.2), (2.1.3). Then

$$ds^2 = \frac{d\theta}{dv} = \frac{d\phi}{dv} = 0, \quad (3.5.1)$$

where v is an affine parameter along these curves. The geodesic equation

$$\frac{d^2 x^s}{dv^2} + \Gamma_{ab}^s \frac{dx^a}{dv} \frac{dx^b}{dv} = 0, \quad (3.5.2)$$

with $x^s = (t, r, \theta, \phi)$, for this type of geodesics has the following non-zero components:

$$\frac{d^2 t}{dv^2} + \Gamma_{rr}^t \left(\frac{dr}{dv} \right)^2 = 0, \quad (3.5.3)$$

$$\frac{d^2 r}{dv^2} + 2\Gamma_{rt}^r \frac{dr}{dv} \frac{dt}{dv} = 0. \quad (3.5.4)$$

After substitution of the Christoffel symbols we obtain

$$\frac{d^2 t}{dv^2} + \dot{a}(t)a(t) \left(\frac{dr}{dv} \right)^2 = 0, \quad (3.5.5)$$

$$\frac{d^2 r}{dv^2} + 2 \frac{\dot{a}(t)}{a(t)} \frac{dr}{dv} \frac{dt}{dv} = 0. \quad (3.5.6)$$

Eq. (3.5.6) can be written in the form

$$\frac{d^2r/dv^2}{dr/dv}dv = -2\frac{\dot{a}(t)}{a(t)}dt, \quad (3.5.7)$$

and integrated out to give

$$\frac{dr}{dv} = \pm \frac{C}{a^2(t)}, \quad (3.5.8)$$

where C is an arbitrary constant and the sign depends on whether the null geodesics are incoming (+), or outgoing (-). Inserting this in Eq. (3.5.5) (with $C = 1$), multiplying by dt/dv and integrating, we find

$$\frac{dt}{dv} = \pm \frac{1}{a(t)}, \quad (3.5.9)$$

from which we choose the minus sign since we are looking for the *past directed* family of null geodesics. The resulting tangent vector field is then

$$k^a = \frac{dx^a}{dv} = \left(-\frac{1}{a(t)}, \pm \frac{1}{a^2(t)}, 0, 0 \right), \quad (3.5.10)$$

which gives

$$g_{ab} \frac{dx^a}{dv} \frac{dx^b}{dv} = - \left(\frac{dt}{dv} \right)^2 + a^2(t) \left(\frac{dr}{dv} \right)^2, = 0 \quad (3.5.11)$$

as desired. The divergence of the family of geodesics is calculated to be

$$k^a_{;a} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} (\sqrt{-g} k^a) = \frac{1}{a^3(t) f^2(r) \sin \theta} \left(\frac{\partial}{\partial t} (-a^2(t) f^2(r) \sin \theta) + \frac{\partial}{\partial r} (\pm a(t) f^2(r) \sin \theta) \right) \quad (3.5.12)$$

i.e.,

$$k^a_{;a} = \frac{2}{a^2(t)} \left(-\dot{a}(t) \pm \frac{f'(r)}{f(r)} \right), \quad (3.5.13)$$

where $f(r) = \sin r, r$, or $\sinh r$ for $k = 1, 0$, or -1 respectively, [91]. A closed trapped surface is then formed if the divergence of both families of null geodesics becomes negative, that is if

$$\dot{a}(t) > \left| \frac{f'(r)}{f(r)} \right|, \quad (3.5.14)$$

for some values of r , t .

An interesting fact for our purposes is that the above condition for the existence of a closed trapped surface has a decisive effect on the electric part $|D|$ of the Bel-Robinson energy. This is seen clearly if we use Eqs. (C.1.9)-(C.1.10) to write $|D|$ in the form

$$|D| = \frac{\sqrt{3}}{a^2(t)} \left| \left(\dot{a}(t) - \frac{f'(r)}{f(r)} \right) \left(\dot{a}(t) + \frac{f'(r)}{f(r)} \right) + \frac{1}{f^2(r)} \right|, \quad (3.5.15)$$

and then use the closed trapped surface condition (3.5.14). In this case, the relation (3.5.14) becomes equivalent to the inequality

$$|D| > \frac{\sqrt{3}}{a^2(t)f^2(r)}, \quad (3.5.16)$$

from which it follows that collapse singularities (as predicted by the existence of a trapped surface) are characterised by a divergent Bel-Robinson energy.

In the next Subsection we illustrate how some of the results found here help to determine the evolution of various model universes.

3.5.2 Examples

Our first three examples illustrate the result that in the case when the CTS condition is combined with the null energy condition in spatially flat universes, an initial or future singularity is predicted to occur and this is reflected in the behaviour of the Bel-Robinson energy. In contrast, our last example 3.5.4 shows that even though the existence of a closed trapped surface is usually associated with the formation of a singularity this need not always be the case. This was recently shown by Ellis in [91] who pointed out that a closed universe can escape the initial singularity provided that the non-spacelike energy condition does not hold. The null energy condition jointly with the condition for the existence of a closed trapped surface are not sufficient to imply a singularity in the case of a closed universe since Theorem 3.5.1 applies *only* in the case of an open or a flat universe.

Example 3.5.1 (Graduated inflation)

Consider the flat graduated inflationary model developed in [92], a universe filled with a fluid with an equation of state $\mu + p = \gamma\mu^{3/4}$ and $\gamma > 0$ (for more details on this family of models see Subsection 4.1.3). The null energy condition¹ is then automatically satisfied. The scale factor for this universe is given by

$$a(t) = \exp\left(-\frac{16}{3^{3/2}\gamma^2 t}\right), \quad (3.5.17)$$

(this is derived by setting $\gamma = 3/4$ in Eq. (4.2.23) of Subsection 4.1.3), and the inequality Eq. (3.5.14) for this case is equivalent to

$$\dot{a}(t) = \frac{16}{3^{3/2}\gamma^2} \frac{1}{t^2} \exp\left(-\frac{16}{3^{3/2}\gamma^2 t}\right) > \frac{1}{r}. \quad (3.5.18)$$

Hence we find that the condition for a CTS becomes valid for large values of r leading to the existence of *past* CTSs. Therefore the assumptions of the singularity theorem 3.5.1 are fulfilled making this model null geodesically incomplete in the past.

Example 3.5.2 (Massless scalar field)

A similar type of singularity is studied in [93] where we meet a flat model filled with a massless scalar field (see Subsection 4.1.1). If we consider a scalar field with $\dot{\phi}^2 \geq 0$ then the null energy condition is satisfied. The inequality (3.5.14) for this case reads

$$\dot{a}(t) = \frac{1}{3t^{2/3}} > \frac{1}{r}, \quad (3.5.19)$$

which is again true for large values of r . Then using the singularity theorem 3.5.1 we conclude that this model is singular, and by the Bel-Robinson energy technique we arrive at a past-collapse singularity of type (S_1, N_1, B_1) .

Example 3.5.3 (Sudden singularity)

¹This condition leads to Eq. (C.1.14) which is equivalent to $\mu + p \geq 0$ for the case of a fluid of density μ and pressure p .

A different situation is described by Barrow in [56] where he considers a model with scale factor

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^n, \quad (3.5.20)$$

with $1 < n < 2$, $0 < q \leq 1$ and t_s arbitrary but fixed from the beginning. This model experiences a sudden singularity (see Definition 3.4.1 ii) and Section 4.4) at the future time t_s with $a(t) \rightarrow a_s$, $\dot{a}(t) \rightarrow \dot{a}_s$ and $\ddot{a}(t) \rightarrow -\infty$. For this model we find that the non-spacelike energy condition² is valid because of $\ddot{a}(t) \rightarrow -\infty$, while for the case of a closed universe there exists a future CTS (the analogous inequality of (3.5.14) is $\dot{a}(t) < -|f'(r)/f(r)|$) since the relation

$$\dot{a}(t) = \frac{q}{t_s^q} t^{q-1} (a_s - 1) + \frac{n}{t_s^n} (t_s - t)^{n-1} < -|\cot r|, \quad (3.5.21)$$

is satisfied for $r = \pi/2$, $a_s < 1$ and $0 < t < t_s$. The singularity theorem 3.5.2 now applies and guarantees that this model is timelike and null geodesically incomplete in the future, while $|E| \rightarrow \infty$ and $|D| < \infty$.

Example 3.5.4 (De Sitter universe)

The possibility of evading the singularity is feasible in a de Sitter universe for which $a(t) = A \cosh Ht$, $k = 1$, $f(r) = \sin r$, so that the existence condition given by Eq. (3.5.14) is satisfied if and only if

$$AH \sinh Ht > |\cot r|, \quad (3.5.22)$$

which holds true for any $t > 0$ and $r = \pi/2$ [91]. This implies the existence of more than one CTSs: every 2-sphere ($t, r = \text{const.}$) with $t > 0$ and area greater than

$$A_s = 4\pi A^2 \cosh^2 Ht \sin^2(r_s), \quad (3.5.23)$$

²As we saw for a null vector this condition is equivalent to $\mu + p \geq 0$. In order to find an analogous inequality for a timelike vector we multiply the Einstein equations by a unit timelike vector k^a : $R_{ab}k^ak^b = (T_{ab} - (1/2)Tg_{ab})k^ak^b = T_{ab}k^ak^b + (1/2)T$. Then $R_{ab}k^ak^b \geq 0 \Leftrightarrow T_{ab}k^ak^b \geq -1/2T$. For a perfect fluid this translates to $\mu + 3p = -6\ddot{a}/a \geq 0$. Therefore the non-spacelike energy condition holds iff $\mu + 3p \geq 0$ and $\mu + p \geq 0$.

where $\cot r_s = AH \sinh Ht$, is a CTS. Nevertheless the spacetime is complete. The timelike energy condition is violated, $(\mu + 3p) = -\ddot{a}(t)/a(t) < 0$, and so Theorem 3.5.2 does not apply. However, the null energy condition (C.1.12) holds true but Theorem 3.5.1 does not apply because the spatial slices are now compact. Then we find that since

$$|E| = \sqrt{3} \left| \frac{\ddot{a}(t)}{a(t)} \right| = \sqrt{3}H^2, \quad (3.5.24)$$

and

$$|D| = \sqrt{3} \left| \frac{\dot{a}^2(t)}{a^2(t)} + \frac{1}{a^2(t)} \right| = \sqrt{3} \left| H^2 \tanh^2 Ht + \frac{1}{A^2 \cosh^2 Ht} \right|, \quad (3.5.25)$$

they always remain finite indicating the complete character of the spacetime in this case (note that the assumptions of the completeness theorem of [82] which we discussed in the end of Section 2.2 are satisfied).

3.6 Discussion

We started off in this chapter by giving a classification of the singularities that can occur in an isotropic universe based on the behaviour of the Hubble parameter. However, such a classification cannot lead to complete results and we have found it necessary to extend this scheme by including succeeding the behaviours of the scale factor and of the Bel-Robinson energy. In this case, we have found that the resulting behaviours of the three functions, H , a and \mathcal{B} taken together exhaust the types of singularities that are possible to form during the evolution of an isotropic universe.

The resulting classification is described by triplets of the form (S_i, N_j, B_l) where the S category monitors the asymptotic behaviour of the expansion rate, closely related to the extrinsic curvature of the spatial slices, the N category that of the scale factor, describing in a sense what the whole of space eventually does, while the B category describes how the various matter fields contribute to the evolution of the geometry on approach to the singularity. We know (cf. [72, 94]) that all these quantities need to be uniformly bounded to produce geodesically complete universes. Otherwise, the whole situation can be very

complicated and we have exploited what can happen in such a case in the relatively simple geometry considered here.

Our scheme not only covers all the recently discovered types of singularities but it also predicts many possible new ones. For example, in the case of a flat isotropic universe the classification of [52] provides us with four main types of singularities. These four types can be identified with four particular (S_i, N_j, B_k) triplets of our scheme: Namely the ‘big rip’ type of [52] characterised by $a \rightarrow \infty$, $\mu \rightarrow \infty$ and $|p| \rightarrow \infty$ at t_s is an (S_2, N_3, B_1) singularity; their ‘sudden’ singularity described by $a \rightarrow a_s < \infty$, $\mu \rightarrow \mu_s < \infty$ and $|p| \rightarrow \infty$ at t_s is an (S_3, N_2, B_3) ; further, their type III singularity, namely $a \rightarrow a_s < \infty$, $\mu \rightarrow \infty$ and $|p| \rightarrow \infty$ at t_s , is clearly an (S_2, N_2, B_1) type singularity, while type IV singularities of [52] with $a \rightarrow a_s < \infty$, $\mu \rightarrow 0$ and $|p| \rightarrow 0$ at t_s , all belong to the (S_3, N_2, B_4) category.

Having all possible singularity types in the unified form expounded in this chapter has the added advantage that we can consistently compare the various different types as we asymptotically approach the time singularity. In this case we saw in detail that the relative *strength* of the functions describing the singularity type becomes an important factor in order to distinguish between all possible behaviours.

It is natural to consider the extension of our classification scheme in the context of more general classes of cosmological spacetimes. We believe that an analysis of this more complicated case is still feasible using the techniques of the present work. In particular, the first step towards that direction would be to examine the family of homogeneous but anisotropic universes. The nature of a singularity arising in such a universe would be given by a triplet analogous to that of the isotropic case but now the number of possible types of singularities will increase. This will result from the existence of more than one scale factors describing the model and also from the fact that the magnetic parts of the Bel-Robinson energy will not be zero as they were in the isotropic case studied here.

A further step in that direction would be to examine inhomogeneous spacetimes where the space gradient of the lapse will be a function not only of the time but also of the *space* variables and hence it will not vanish automatically. The nature of a singularity would

then be given by a quadruplet with the new component describing the possible behaviours of the space gradient of the lapse. These behaviours are predicted from the contrapositive of the completeness theorem 2.2.1 and they include both a diverging $|\nabla N|_g$ in a finite time (giving rise to a possible *blow-up lapse singularity*) as well as a $|\nabla N|_g$ which will be finite only for a finite interval of proper time corresponding to a *sudden lapse singularity*. The analysis of all these cases lies beyond the work in the present thesis.

Chapter 4

Singularities in isotropic cosmologies

4.1 Introduction

In the previous chapter we derived a complete classification of spacetime singularities that are in principle possible to arise in isotropic cosmologies. The purpose of this chapter is to look for physical circumstances that give rise to four particularly important members of our classification given by Definition 3.4.1; ideally we would be content with prototypes of cosmologies with every possible singularity type that is predicted by our classification scheme.

The analysis of the various cosmologies appearing in this chapter is organised as follows. In Section 4.2 we study collapse singularities that arise in recently studied cosmologies containing a massless scalar field (Subsection 4.2.1), a scalar field with an exponential potential (Subsection 4.2.2), or a graduated inflationary potential (Subsection 4.2.3). Section 4.3 provides an analysis of big-rip singularities which are met in models containing dark energy in the form of phantom fields (Subsection 4.3.1), or Chaplygin gases (Subsection 4.3.2). This type of singularity offers a new possibility of a violent end for the universe towards which every bound existing structure gets ‘ripped apart’. In Section 4.4, we explore the character of sudden singularities that may arise in cosmologies filled with a fluid having either *unconnected* pressure and density (Subsection 4.4.1), or alternatively, obey-

ing an equation of state (Subsection 4.4.2). In Subsection 4.4.3, we analyse models that exhibit milder than sudden singularities. Finally, in Section 4.5, we discuss sudden-rip type singularities.

In each one of these sections we prove theorems giving necessary and sufficient conditions for the occurrence of the various types of singularity that are analysed. We expect the proofs of all these theorems to be quite straightforward for we have now *already* identified the type of singularity that we are looking for in accordance to our classification. Proving such theorems *without* this knowledge would have been a problem of quite a different order.

The usefulness of these theorems lies in that they answer the question of whether or not the behaviours met in known cosmological models described by exact solutions (which as a rule have a *flat* spatial metric ($k = 0$)) continue to be valid in universes having nonzero values of k , or described by solutions which are more general than being exact in the sense that some or all of the arbitrary constants present in a solution remain arbitrary. We will see that whenever the behaviour met in flat models passes over to curved ones either the curvature term in the Friedman equation turns out to be subdominant compared to the density term, or, in any case, cannot significantly alter the behaviour of the Hubble parameter H .

It is worth mentioning that all the different example cases of cosmological models analysed in this Chapter have been a primary source of motivation for finding more general theoretical criteria which determine their evolution and mark their basic characteristics.

4.2 Collapse singularities

Following the Definition 3.4.1 given in the previous chapter, by past or future collapse singularities we mean those that arise in solutions with a scale factor tending to zero at an initial time (big-bang time), or within a finite future time (big-crunch time) respectively, while the Hubble expansion rate and the electric parts of the Bel-Robinson energy diverge. Originally, collapse singularities were discovered in simple FRW universes filled with dust or radiation (cf. [77], pp. 96-101). Here, we will instead focus on *recently* analysed models

that involve the presence of a scalar field (cf. [93], [95], [92], [96]).

Scalar fields play a substantial role in cosmology. In the framework of inflation, scalar fields having a self-interaction potential are used to model the matter content of the early universe and their dynamical evolution monitors the mechanism of inflation (cf., e.g., [97], pp. 261-317). However, the role of scalar fields is not limited to early universe scenarios. They are also crucial in the present phase of evolution of the universe since scalar fields actualise a possible dark-energy component that is believed to drive the presently observed accelerating expansion of the universe (cf. [51]). Furthermore, in grand unified theories scalar fields are met in the spontaneous symmetry breaking (cf. [97], pp. 195-255, [98]) and other fundamental mechanisms proposed within these theories (cf. [97]).

In a homogeneous universe, scalar fields are functions of the proper time only. The energy-momentum tensor of a scalar field is equivalent to that of a perfect fluid with density μ_ϕ and pressure p_ϕ given by (cf. Appendix D),

$$\mu_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \quad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \quad (4.2.1)$$

The relation between the pressure and the energy density of the scalar field is therefore

$$p_\phi = \mu_\phi - 2V,$$

which is not an equation of state in the usual sense [99].

However, in the special case of a massless scalar field, $V(\phi) = 0$, we have an equation of state of the form $p_\phi = \mu_\phi$, that is it behaves as stiff matter. If on the other hand, the energy of the scalar field is purely potential, or dominates over the kinetic energy, then the scalar field is approximately described by the equation of state $p_\phi = -\mu_\phi$, and behaves therefore like a cosmological constant. Within the framework of inflationary cosmology for example, the energy of the scalar field becomes *mainly* potential as soon as the field enters the slow-roll phase. This means that $\dot{\phi}^2 \ll V(\phi)$, or equivalently $\mu_\phi + 3p_\phi < 0$, and hence the dominant energy condition is violated in this case. Therefore the singularity theorem 3.5.1

does not apply and the initial singularity is not an inevitable consequence of inflationary cosmologies. Nevertheless, there exist interesting, non-trivial scalar field cosmologies that cannot avoid the initial singularity in this way and in the next subsections we shall focus on the analysis of such cosmologies.

4.2.1 Massless scalar field

An example of a singular scalar field cosmology that possesses a past-collapse singularity can be found in [93]. This model is a flat FRW universe with a massless scalar field, and the Friedman and continuity equations read

$$3H^2 = \frac{1}{2}\dot{\phi}^2, \quad \ddot{\phi} + 3H\dot{\phi} = 0, \quad (4.2.2)$$

from which we readily find:

$$H = \frac{1}{3t}, \quad \phi = \pm\sqrt{\frac{2}{3}} \ln \frac{t}{c}. \quad (4.2.3)$$

Indeed, this universe emerges from a past-collapse singularity at $t = 0$, since $a \propto t^{1/3} \rightarrow 0$, $H \rightarrow \infty$, while $|E|^2 = \dot{\phi}^4/3 \rightarrow \infty$ and $|D|^2 = 3H^4 \rightarrow \infty$.

In [93] Foster also found a condition that should be satisfied so that solution (4.2.3), approximates that with an arbitrary potential $V(\phi)$ in the limit $t \rightarrow 0$. This condition is derived by simply substituting the solution given by Eq. (4.2.3) in the Friedmann equation that is satisfied by a scalar field with potential $V(\phi)$, i.e. $3H^2 = V(\phi) + \dot{\phi}^2/2$, and equating the resulting terms. One then finds that

$$\lim_{t \rightarrow 0} t^2 V \left(\pm\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right) = 0, \quad (4.2.4)$$

or

$$\lim_{\phi \rightarrow \pm\infty} e^{-\sqrt{6}|\phi|} V(\phi) = 0. \quad (4.2.5)$$

Hence the solution given by Eq. (4.2.3) approximates the solution for a scalar field with potential $V(\phi)$ when the potential decreases slower than $e^{-\sqrt{6}|\phi|}$. The following theorem

implies that this type of singularity may also arise in more general, curved universes [69], [70].

Theorem 4.2.1 *A necessary and sufficient condition for the occurrence of a past-collapse singularity of type (S_1, N_1, B_1) at time t_1 in an isotropic universe with a massless scalar field is that $\dot{\phi} \rightarrow \infty$ at t_1 .*

PROOF. We first prove necessity. Integrating the continuity equation $\ddot{\phi} + 3H\dot{\phi} = 0$, we find that $\dot{\phi} \propto a^{-3}$. Since $\dot{\phi} \rightarrow \infty$, we have that $a \rightarrow 0$. Because

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2} \rightarrow \infty, \quad \text{as } t \rightarrow t_1, \quad (4.2.6)$$

H becomes unbounded at t_1 . In addition, since

$$|D|^2 = \frac{\mu^2}{3} = \frac{\dot{\phi}^4}{12} \rightarrow \infty, \quad (4.2.7)$$

and

$$|E|^2 = \frac{1}{12}(\mu + 3p)^2 = \frac{\dot{\phi}^4}{3} \rightarrow \infty, \quad (4.2.8)$$

as $t \rightarrow t_1$, both $|D|$ and $|E|$ diverge there. Conversely, assuming a past-collapse singularity of type (S_1, N_1, B_1) at t_1 , we have from condition B_1 ($\Leftrightarrow \mu \rightarrow \infty, |\mu + 3p| \rightarrow \infty$) that $\mu \rightarrow \infty$ and so $\dot{\phi}^2 \rightarrow \infty$ as $t \rightarrow t_1$. ■

It follows that the asymptotic strength of the singularity in this model (described by both the exact solution and the above theorem) is given by: $a \ll H \ll (|E| \sim |D|)$.

4.2.2 Multiple exponential potential

Since there is not as yet any unique well-established fundamental theory dictating and predicting an exact form of the potential $V(\phi)$, there are many possible candidate potentials currently studied depending on the underlying theory they intend to describe. For example, in [95] the authors consider a scalar field with the ‘multiple’ exponential potential of the

form

$$V(\phi) = W_0 - V_0 \sinh(\sqrt{3/2}\phi),$$

where W_0 and V_0 are arbitrary constants. The scalar field inhabits a flat FRW universe containing also a separately conserved pressureless fluid. In this context, the field equations become

$$3H^2 = \left(\frac{\dot{\phi}}{2} + V(\phi) + \mu \right), \quad (4.2.9)$$

$$2\dot{H} + 3H^2 = - \left(\frac{\dot{\phi}}{2} - V(\phi) \right), \quad (4.2.10)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (4.2.11)$$

$$\dot{\mu} + 3H\mu = 0. \quad (4.2.12)$$

Following [95], we split the scale factor in a product form by setting $a^3 = xy$, while the scalar field is written as $\phi = \sqrt{2/3} \log(y/x)$ with x and y positive. Setting

$$x = C[\exp \chi_1 \cos \chi_2 + \exp(-\chi_1) \cos \chi_2], \quad (4.2.13)$$

$$y = C[\exp \chi_1 \sin \chi_2 + \exp(-\chi_1) \sin(-\chi_2)] \quad (4.2.14)$$

(here, $\chi_1 = w_1(t - t_0)$, $\chi_2 = w_2(t - T_0)$, $C > 0$, t_0 is an arbitrary constant, T_0 is the ‘initial’ time and w_1, w_2 positive parameters such that $w_1^2 - w_2^2 = 3/4W_0$, $2w_1w_2 = 3/4V_0$), we find that for large and positive values of the time parameter t the scale factor becomes

$$a = C^{2/3} \exp(2/3\chi_1) \cos^{1/3} \chi_2 \sin^{1/3} \chi_2, \quad (4.2.15)$$

which is obviously divergent.

At the finite time value $T_s = \pi/(2w_2) + T_0$, we have that $a \rightarrow 0$, $H \rightarrow w_2(\cot \chi_2 - \tan \chi_2)/3 \rightarrow \infty$, and consequently we find that

$$|D| \rightarrow \infty, \quad (4.2.16)$$

and also

$$|E| \rightarrow |2w_2(2(\cot \chi_2 - \tan \chi_2) - \frac{w_2}{9}(\cot^2 \chi_2 + \tan^2 \chi_2))| \rightarrow \infty. \quad (4.2.17)$$

This model therefore serves as a nice example that illustrates the structure of the future-collapse type singularity predicted earlier in our classification.

4.2.3 Graduated inflation

In the original inflationary scenario the equation of state $p = -\mu$ that describes the scalar field during the slow-roll period is eventually responsible for an accelerating exponential expansion. However, milder forms of solutions, for example a power-law expansion $a \propto t^p$ with $p > 1$, can still suffice for inflation. Different forms of solutions in which the expansion follows a power-law with an arbitrary in general exponent lead to the so-called *intermediate* or *graduated-inflationary* universes (cf. [92] and references therein). These models were first studied by Barrow who introduced an equation of state of the form

$$\mu + p = \gamma\mu^\lambda, \quad (4.2.18)$$

where λ, γ are constants and $\gamma \neq 0$, in a flat model. The set of solutions that Barrow found (which naturally depended on the λ and γ parameters) included not only the traditional power-law and exponential inflationary solutions but also new ones. In these solutions the scale factor is an exponential of an arbitrary power of the time and in some cases describes a universe that begins with a past-collapse singularity. In the original references, however, no complete discussion of the structure of singularities predicted in these models is included and so we give below such an analysis.

To see how a past-collapse singularity develops in the history of these cosmologies we substitute the equation of state (4.2.18) in the continuity equation and integrate to find

$$\mu = \frac{(3\gamma(\lambda - 1))^{1/(1-\lambda)}}{(\ln(a/a_0))^{1/(\lambda-1)}}, \quad (4.2.19)$$

where $\lambda \neq 1$ and a_0 is a constant. Using then the Friedmann equation $3H^2 = \mu$, we obtain the scale factor in the following explicit form [92]:

$$\ln\left(\frac{a}{a_0}\right) = \frac{3^{\lambda/(1-2\lambda)}\gamma^{1/(1-2\lambda)}(\lambda - 1/2)^{2(1-\lambda)/(1-2\lambda)}}{\lambda - 1} t^{2(1-\lambda)/(1-2\lambda)}, \quad (4.2.20)$$

with $\lambda \neq 1, 1/2$. Setting

$$A = \frac{3^{\lambda/(1-2\lambda)}\gamma^{1/(1-2\lambda)}(\lambda - 1/2)^{2(1-\lambda)/(1-2\lambda)}}{\lambda - 1}, \quad (4.2.21)$$

$$f = 2(1 - \lambda)/(1 - 2\lambda), \quad (4.2.22)$$

we can write the scale factor in the form

$$a(t) \propto \exp(At^f). \quad (4.2.23)$$

Considering the case that $1/2 < \lambda < 1$ and $\gamma > 0$, we have that $A < 0$ and $f < 0$. This means that a past-collapse singularity develops as $t \rightarrow 0$, since then $a \rightarrow 0$, $H = Aft^{f-1} \rightarrow \infty$ and consequently

$$|D|^2 = 3H^4 \rightarrow \infty, \quad (4.2.24)$$

and also $\mu \rightarrow \infty$, which in turn implies that

$$|E|^2 = \frac{1}{12}(-2\mu + 3\gamma\mu^\lambda)^2 \rightarrow \infty. \quad (4.2.25)$$

The asymptotic strength of this singularity is therefore: $a \ll (|E| \leftrightarrow |D|) \ll H$. Actually this type of behaviour is found to occur more generally due to the following result.

Theorem 4.2.2 *A necessary and sufficient condition for a past-collapse singularity (S_1, N_1, B_1) to occur at time t_1 in an open or flat universe filled with a fluid with equation of state $p + \mu = \gamma\mu^\lambda$, $\gamma > 0$ and $\lambda < 1$, is that $\mu \rightarrow \infty$ as $t \rightarrow t_1$.*

PROOF. The continuity equation Eq. (2.1.6) gives directly the form of the scale factor,

$$a = a_0 \exp\left(\frac{\mu^{-\lambda+1}}{3\gamma(\lambda-1)}\right), \quad (4.2.26)$$

so that $a \rightarrow 0$ as $t \rightarrow t_1$. Since

$$\begin{aligned} H^2 &= \frac{\mu}{3} - \frac{k}{a^2} = \frac{\mu}{3} - k a_0^{-2} \exp\left(\frac{-2\mu^{-\lambda+1}}{3\gamma(\lambda-1)}\right) > 0, \\ |E|^2 &= \frac{1}{12}(-2\mu + 3\gamma\mu^\lambda)^2, \\ |D|^2 &= \frac{\mu^2}{3}, \end{aligned}$$

we see that as $t \rightarrow t_1$ H , $|E|$ and $|D|$ diverge provided that $k = 0$ or $k = -1$. The converse is straightforward. ■

As it was pointed out in [92] the matter content in a graduated inflationary universe can be reinterpreted as a scalar field with the following potential:

$$V(\phi) = 3 \left(Af \left(\frac{f-1}{8A(1-f)} \right)^{(f-1)/f} \phi^{2(f-1)/f} \right)^2 - Af(1-f) \left(\frac{f-1}{8A(1-f)} \right)^{(f-2)/f} \phi^{2(f-2)/f}, \quad (4.2.27)$$

where

$$\phi = \left(\frac{8A(1-f)}{f} \right)^{1/2}. \quad (4.2.28)$$

and where f is given by Eq. (4.2.22). It is worth noting here that it is only recently that this model has regained interest since, as it was shown in [96], a specific choice for the f parameter offers a good fit to the WMAP observations [55] within the framework of the slow-roll approximation.

In order to illustrate this result, let us briefly review the equations of the slow-roll approximation. For large ϕ the potential (4.2.27) becomes [96]

$$V(\phi) = \frac{48A^2}{(\beta+4)^2} (2A\beta)^{\beta/2} \phi^{-\beta}, \quad (4.2.29)$$

where $\phi = (2A\beta t^f)^{1/2}$ and $\beta = 4(f^{-1} - 1)$. In the slow-roll approximation it is assumed

that $\dot{\phi}^2 \ll V$ and $\ddot{\phi} \ll V'$ restricting the field equations to the form (cf. [100], [101])

$$3H^2 \simeq V \quad (4.2.30)$$

$$3H\dot{\phi} \simeq -V'. \quad (4.2.31)$$

Inflation then occurs if

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \Leftrightarrow -\frac{\dot{H}}{H^2} < 1. \quad (4.2.32)$$

Differentiating Eq. (4.2.30) with respect to t and using Eq. (4.2.31) we find that

$$\dot{H} = -\frac{(V')^2}{18H^2}, \quad (4.2.33)$$

so that

$$-\frac{\dot{H}}{H^2} = \frac{1}{2} \left(\frac{V'}{V} \right)^2, \quad (4.2.34)$$

while condition (4.2.32) holds if and only if the *first slow-roll parameter* ϵ , defined by [100], [101]

$$\epsilon \equiv \frac{1}{2} \left(\frac{V'}{V} \right)^2, \quad (4.2.35)$$

satisfies the condition

$$\epsilon \ll 1. \quad (4.2.36)$$

If we further differentiate Eq. (4.2.31) and use Eq. (4.2.36) together with $\ddot{\phi} \ll V'$, we conclude that the *second slow-roll parameter* η , defined by [100], [101]

$$\eta \equiv \frac{V''}{V}, \quad (4.2.37)$$

satisfies the condition

$$|\eta| \ll 1. \quad (4.2.38)$$

However, as it is stated in [102], it is not necessary to demand that the derivative of an approximation equation is *itself* a valid approximation equation and so this last condi-

tion can be regarded as independent from the first one given by Eq. (4.2.36). In the intermediate-inflationary model of [96] these parameters are shown to be $\epsilon = \beta^2/2\phi^2$ and $\eta = \beta(1 + \beta/2)/\phi^2$. As it is further shown in [96], the resulting values of two very important spectral indices that are closely related to these slow-roll parameters come in good agreement with their corresponding values measured by observations, provided that $\beta = 2$, i.e. $f = 2/3$ and hence $a(t) = \exp(At^{2/3})$. A graduated-inflationary model with these parameters and scale factor offers therefore an appropriate theoretical framework within which the data from the WMAP observations is explained.

4.3 Big-rip singularities

Let us now move on to the discussion of a less standard type of model, nevertheless one which is singular in a way predicted by our scheme. The notion of dark matter was introduced in cosmology in order to explain a series of observations in the area of galactic dynamics (cf. [103] pp. 417-424). These observations showed that the velocities of galaxies in clusters are so high and the luminous masses enclosed within the galaxies so low that these structures ought to have been ripped apart. A similar situation holds for the case of spiral galaxies. In particular, it was found in [104] that at a distance r outside the central luminous part of such a galaxy the rotational velocity $v(r)$ does not follow the anticipated third law of Kepler $v(r) \propto r^{-1/2}$ but is instead measured to be constant. This leads to the counter-intuitive fact that the mass at the outskirts of such a galaxy grows proportionally to distance (cf. [103] pp. 46-47). Such a pattern of matter distribution could be attributed to the existence of a halo surrounding these galaxies and consisting of low-luminosity matter, thereby called dark matter, that contributes the missing amount of matter needed to fit the observed profile of these galaxies (see [103] pp. 45-53).

Dark matter is also considered in another context. Observations strongly indicate that the value of Ω , the ratio of the present density of the universe to the critical density $\mu_c = 3H_0^2/8\pi G$ (H_0 being the present value of H), is unity. However, the luminous mass of the universe is currently measured to have a value of Ω_{Lum} only around 0.04 [105]. On

the other hand, measurements of the amount of dark matter that is believed to exist in clustered areas of the universe give $\Omega_{Dm} \simeq 0.26$ [105]. This missing part, nearly 70% of the critical density, is contributed by an unclustered component the dark energy which has a ratio of pressure to density $w = p/\mu < -0.78$ [105]. Taking this into account we see that dark matter and dark energy may play an important role in the understanding of the composition of the universe. Even though the nature of dark energy remains largely a mystery it is an issue worth examining as it offers us the opportunity to discover new and exciting possible fates of model universes with such forms of material content.

The simplest source of dark energy can be described by simply postulating the existence of a cosmological constant Λ which satisfies a perfect fluid equation of state, $p = w\mu$ with $w = -1$ [106]. However, the consideration of a cosmological constant Λ in this context leads to the so-called cosmological constant problem - the *expected* value of the cosmological constant from quantum field theories is some 120 orders of magnitude larger than the observed one [3]. Another candidate for dark energy is the *quintessence*, a scalar field which satisfies the equation of state for $-1 < w < -1/3$ (cf. [107]-[113]). Because recent observations indicate that the universe is not only expanding but is also accelerating a lot of attention has been drawn to use such matter components (or even more exotic ones, cf. [2]-[47]) in an effort to explain the observed acceleration [54] [55]. Note that in order to obtain an accelerating FRW universe filled with a perfect fluid equation of state $p = w\mu$, it is necessary to have a value of the w parameter that is less than $-1/3$ as this clearly follows from the field equations (cf. 4.3.3) below).

Cosmological observations do not exclude the fact that the w parameter be less than -1 [3]. An entity satisfying such an exotic equation of state is dubbed *phantom dark energy*; this is a becoming name for the description of something that is physically unorthodox but which nevertheless seems to provide an explanation to the observations [53]. To begin with, phantom dark energy immediately violates the dominant energy condition since $p + \mu = \mu(w + 1) < 0$. This leaves room for the existence of wormholes and times machines (see [3] for a discussion). Another interesting albeit unphysical property of phantom dark energy

is that it exhibits a sound speed that it is both imaginary and superluminal [2].

Despite the strange properties of phantom-dark energy it is still an interesting type of matter field to study in cosmology (see Refs. [2]-[47], [53], for phantom filled cosmological models) since its presence influences the future of the universe through the development of a big-rip singularity as is shown in the next Subsection. In what follows, we will use the term *phantom cosmology* to refer to any model that includes a phantom-matter component.

4.3.1 Phantom dark energy

Consider the case of a flat FRW universe with metric $ds^2 = -dt^2 + a^2(t)d\bar{x}^2$, filled with a fluid with equation of state $p = w\mu$. The Einstein field equations for such a universe are

$$\mu = 3H^2, \quad (4.3.1)$$

$$\dot{\mu} + 3H\mu(1+w) = 0, \quad (4.3.2)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\mu(1+3w). \quad (4.3.3)$$

Integrating the continuity equation we have

$$\mu = Ca^{-3(w+1)}, \quad (4.3.4)$$

where C is the integration constant. For ordinary matter ($w \geq 0$), or for dark matter with $w > -1$, the energy density decreases as the universe expands. In contrast, phantom dark energy has a density that *increases* as the expansion proceeds in an FRW universe (irrespectively of its spatial geometry). This implies that it will at first overcome and finally dominate over all components of ordinary matter.

Solving the Friedmann equation (4.3.1) for a , we find [51]

$$a = \left[a_0^{3(1+w)/2} + \frac{3(1+w)\sqrt{A}}{2}(t-t_0) \right]^{\frac{2}{3(1+w)}}, \quad (4.3.5)$$

where $A = C/3$ and a_0 is an integration constant. The behaviour that would have been expected had we considered only ordinary matter is that $a \rightarrow \infty$ as $t \rightarrow \infty$, and the Hubble distance H^{-1} to grow more rapidly than the scale factor (as it follows from Eq. (4.3.5) and Eq. (4.3.7) below), which means that more and more galaxies would become visible asymptotically [51]. In contrast to this situation, in a phantom dark energy model a big-rip singularity arises within the finite value of time given by [51]

$$t_{rip} = t_0 + \frac{2}{3\sqrt{A}(|w| - 1)a_0^{3(|w|-1)/2}}, \quad (4.3.6)$$

and called the *big-rip time*. Note that t_{rip} depends on the initial size of the universe, a_0 , and the exact value of the w parameter in such a way that the bigger the values of a_0 and w , the briefer the existence of the phantom universe. Indeed, from Eq. (4.3.5) we see that the scale factor becomes divergent as $t \rightarrow t_{rip}$. Also the Hubble rate and hence the norms the electric parts of the Bel-Robinson energy become diverge there since we can obtain the following forms:

$$H^2 = A \left[a_0^{3(1+w)/2} + \frac{3(1+w)\sqrt{A}}{2}(t - t_0) \right]^{-2}, \quad (4.3.7)$$

$$|E|^2 = \frac{3}{4}H^4(1 + 3w)^2, \quad (4.3.8)$$

$$|D|^2 = 3H^4. \quad (4.3.9)$$

Eq. (4.3.7) implies in particular that the Hubble *distance* H^{-1} now tends to zero as $t \rightarrow t_{rip}$. Combining this with the infinite growth of the scale factor as $t \rightarrow t_{rip}$, we see that this type of accelerated expansion forces galaxies to disappear beyond the cosmological horizon rendering the end of the universe dark and rather violent. The latter attribute follows from the fact that it can be shown (see [3]) that as the big-rip time approaches gravitationally bound structures such as galaxies, planetary systems or even atoms, become unbound due to the enormous repulsive pressure.

The following theorem [69], [70] characterises the future singularity in phantom cos-

mologies *irrespective* of the value of the curvature k and implies that the singularities in such models can be milder than the standard all-encompassing future-collapse singularities and have necessarily diverging pressure. This theorem generalises and unifies previous results (cf. [3], [48], [49], [50], [51], [52], [53], [14], [38]).

Theorem 4.3.1 *Necessary and sufficient conditions for a big-rip singularity (S_2, N_3, B_1) to occur at the finite future time t_s in a universe filled with a fluid with equation of state $p = w\mu$, are that $w < -1$ and $|p| \rightarrow \infty$ at t_s .*

PROOF. Substituting the equation of state $p = w\mu$ in the continuity equation $\dot{\mu} + 3H(\mu + p) = 0$, we have

$$\mu \propto a^{-3(w+1)}, \quad (4.3.10)$$

and so if $w < -1$ and p blows up at t_s , a also blows up at t_s . Since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2}, \quad |D|^2 = \frac{\mu^2}{3}, \quad |E|^2 = \frac{1}{12}\mu^2(1 + 3w)^2, \quad (4.3.11)$$

we conclude that at t_s , H , a , $|D|$ and $|E|$ are divergent. Conversely, assuming a big-rip singularity (S_2, N_3, B_1) at t_s in an FRW universe with the equation of state $p = w\mu$, we have from the (B_1) hypothesis that $\mu \rightarrow \infty$ at t_s and so p also blows up at t_s . Since a is divergent as well, we see from (4.3.10) that $w < -1$. ■

It is interesting to note that the relative strength of this big-rip singularity is strongly influenced by the choice of the w parameter. In particular, there are three possible behaviours depending on the given ranges of the w parameter:

- if $-4/3 < w < -1$, then $H \ll (|E| \sim |D|) \ll a$,
- if $-5/3 < w < -4/3$, we have $H \ll a \ll (|E| \sim |D|)$, whereas
- if $w < -5/3$, we have $a \ll H \ll (|E| \sim |D|)$.

We note that for all three ranges of w , $|E|$ and $|D|$ diverge similarly while H remains always much smaller compared to these two quantities. On the other hand, the scale factor diverges less strongly relatively to H and $|E|$, $|D|$ as w decreases below -1 .

4.3.2 Chaplygin gas

As we saw in the previous Section a phantom equation of state drives the universe towards a big-rip singularity. In this section we will see that a big-rip singularity can still arise in models with forms of dark energy different from phantoms such as the so-called *generalised* Chaplygin gases; these fields play a central role in dark energy cosmology and specifically in the unification of dark energy and dark matter (see Refs. [114]-[138]).

For example, consider a flat FRW universe containing a dark-energy component which at late times satisfies a general equation of state of the form [48]

$$p = -\mu_d - A\mu_d^\alpha \quad (4.3.12)$$

where A and α are real parameters. Integrating the continuity equation

$$\dot{\mu}_d - 3HA\mu_d^\alpha = 0, \quad (4.3.13)$$

we find that the energy density μ_d of the dark component behaves as follows,

$$\mu_d = \mu_{d0} \left(1 + 3\bar{A}(1 - \alpha) \ln \frac{a}{a_0} \right)^{\frac{1}{1-\alpha}}, \quad (4.3.14)$$

where $\bar{A} = A\mu_{d0}^{\alpha-1} > 0$ and μ_{d0} is an integration constant. Note that

$$\frac{d\mu_d}{da} = \frac{3\bar{A}\mu_{d0}}{a} \left(1 + 3\bar{A}(1 - \alpha) \ln \frac{a}{a_0} \right)^{\frac{1}{1-\alpha}-1}, \quad (4.3.15)$$

so that the parameter \bar{A} monitors the behaviour of the dark energy as the universe expands: if $\bar{A} > 0$ the density of the dark energy increases with expansion if $\bar{A} = 0$, the density is constant, whereas if $\bar{A} < 0$, the density decreases [48]. Recall that for the occurrence of a big-rip singularity the density needs to grow with expansion and hence for the study of such a singularity we may focus here in the case $\bar{A} > 0$.

Substituting in the Friedmann equation (2.1.4) we find the scale factor in the following

implicit form

$$\left(1 + 3\bar{A}(1 - \alpha) \ln \frac{a}{a_0}\right)^{\frac{1-2\alpha}{2(1-\alpha)}} - \left(1 + 3\bar{A}(1 - \alpha) \ln \frac{a_d}{a_0}\right)^{\frac{1-2\alpha}{2(1-\alpha)}} = \frac{3}{2}\bar{A}(1 - 2\alpha)C(t - t_d). \quad (4.3.16)$$

Here $B = \sqrt{\mu_{d0}/3}$, t_d is the time at which the dark energy becomes dominant and a_d is the corresponding value of the scale factor $a_d = a(t_d)$. The Hubble expansion rate is then given by

$$H = B \left(\left(1 + 3\bar{A}(1 - \alpha) \ln \frac{a_d}{a_0}\right)^{\frac{1-2\alpha}{2(1-\alpha)}} + \frac{3}{2}\bar{A}(1 - 2\alpha)C(t - t_d) \right)^{\frac{1}{1-2\alpha}}. \quad (4.3.17)$$

We now show that the future behaviour of this model is quite sensitive to the values of the parameter α . Although the case with $\alpha \in (1/2, 1)$ leads to the development of a big-rip singularity in finite time, when $\alpha < 1/2$ we are led instead a singularity-free universe. Indeed, taking $\alpha \in (1/2, 1)$ we allow for the dark energy density, the scale factor and the Hubble rate H all to increase with time until they diverge at the finite time t_f where

$$t_f = t_d + \frac{2}{3\bar{A}(2\alpha - 1)C} \left(1 + 3\bar{A}(1 - \alpha) \ln \frac{a_d}{a_0}\right)^{\frac{1-2\alpha}{2(1-\alpha)}}. \quad (4.3.18)$$

In addition, the Bel-Robinson energy blows up there since the electric parts take the forms:

$$|E|^2 = \frac{1}{12}(2\mu + 3A\mu^a)^2, \quad (4.3.19)$$

$$|D|^2 = 3H^4. \quad (4.3.20)$$

If instead one assumes that $\alpha < 1/2$, we see from Eq. (4.3.17) that H is always finite (so that the Bel-Robinson energy is also finite) and therefore the model is geodesically complete. (The case with $\alpha > 1$ leads to an (S_2, N_2, B_1) singularity since as it follows from (4.3.14) μ_d and consequently p_d , H , $|E|$ and $|D|$ all diverge at the finite value $a_s = a_0 e^{1/(3\bar{A}(\alpha-1))}$.)

Summing up the results of this Section, the consideration of these unconventional dark

energy components in the framework of FRW cosmology leads to the possibility of a big-rip singularity. Even though Theorem 3.2.1 predicts such a singular behaviour in condition S_2 , it is only very recently that the study of physically motivated models described here has offered sufficient *physical* grounds for the appearance of such a structure distinct from the standard future-collapse types considered so far (cf. [3], [48], [49], [50], [51], [52], [53], [14], [38]).

4.4 Sudden singularities

In the previous Section, we saw that the violation of the dominant energy condition is sufficient for the occurrence of future finite-time singularities during the expansion of the universe. But is this also necessary? Can a finite-time singularity develop during the expansion of the universe *without* violating the dominant energy condition? If yes, then what are the qualitative differences between the resulting types of singularities?

Barrow initiated a study of these questions in [56]. He found a new type of singularity that can arise in a universe where matter does not violate the dominant energy condition. Actually in his example matter satisfies even the strong-energy conditions $\mu > 0$ and $\mu + 3p > 0$. This fact implies that the violation of the dominant-energy condition is sufficient but *not* necessary for the occurrence of a finite-time singularity during expansion. Moreover, the nature of the resulting singularity is substantially different from the big-rip singularities discussed earlier. In particular, the finite-time singularity of Barrow is a sudden singularity according to our Definition 3.4.1 occurring at an arbitrary but fixed value of finite time t_s at which the scale factor, the Hubble rate and consequently $|D|$, all remain finite but $|E|$, or equivalently the acceleration, diverges. This becomes possible since by the Raychaudhuri equation at t_s we have

$$\frac{\ddot{a}}{a} = -\frac{p}{2} - \frac{H_s^2}{2} - \frac{k}{2a_s^2}, \quad (4.4.1)$$

where $a_s \equiv a(t_s)$, $H_s \equiv H(t_s) > 0$, which implies that a singularity with $\ddot{a}/a \rightarrow -\infty$ can

occur provided the pressure blows up at t_s .

In the following Subsections we analyse how a sudden singularity arises in universes filled with a fluid having μ and p unconnected (Subsection 4.4.1), or satisfying an equation of state (Subsection 4.4.2). In Subsection 4.4.3 we explore model universes that exhibit singularities that are milder than the sudden ones.

4.4.1 Fluids with p and μ unconnected

A special exact solution acquiring all the characteristics needed for the development of a sudden singularity is given in [56] and reads

$$a(t) = 1 + \left(\frac{t}{t_s}\right)^q (a_s - 1) + \left(1 - \frac{t}{t_s}\right)^n, \quad (4.4.2)$$

with $1 < n < 2, 0 < q \leq 1$. For the more detailed analysis of the properties of this solution given below we follow [67], [68]. This solution exists smoothly only on the interval $(0, t_s)$, with a_s and H_s and therefore $|D|$ is finite at the right end. By setting $A = q(a_s - 1)/t_s^q, B = n/t_s^n$, we find that

$$\dot{a}(t) = At^{q-1} + B(t_s - t)^{n-1}, \quad (4.4.3)$$

which means that unless $q = 1$, \dot{a} blows up as $t \rightarrow 0$, making H continuous only on $(0, t_s)$. Also $a(0)$ is finite and we can extend H and define it to be finite also at 0, $H(0) \equiv H_0$, so that H is defined on $[0, t_s]$. However, since $\lim_{t \rightarrow 0^+} H(t) = \pm\infty$, H is non-integrable on $[0, t_s]$, t_s arbitrary and thus this model universe implements exactly Condition $S1$ of Theorem 3.2.1.

Another basic feature of this solution is that the second derivative of the scale factor is

$$\ddot{a}(t) = q(q-1)Bt^{q-2} - \frac{n(n-1)}{t_s^2(1-t/t_s)^{2-n}} \rightarrow -\infty, \quad \text{as } t \rightarrow t_s, \quad (4.4.4)$$

so that $|E|$, and consequently p , diverges at t_s as it is expected for a sudden singularity. Note that $\mu_s > 0$ for $q^2 t_s^{-2} (a_s - 1)^2 > -k$, and since $(\mu + 3p) = -6\ddot{a}/a$ (see Eq. (2.1.5)), we can have $\mu > 0$ and $\mu + 3p > 0$ throughout the evolution of such an expanding universe.

We can further prove that under the additional assumption $p \geq 0$, a universe exhibiting a sudden singularity is geodesically incomplete [67]. Firstly, we know (cf. [1], pp. 141-2) that in an FRW universe with $\mu > 0$ and $p \geq 0$, given any vector X at any point q , the geodesic $\gamma(v)$ which passes through the point $q = \gamma(0)$ in the direction of X is such that

- either $\gamma(v)$ can be extended to arbitrary values of its affine parameter v , or
- there is some value $v_0 > 0$ such that the scalar invariant $(R_{\alpha\beta} - 1/2Rg_{\alpha\beta})(R^{\alpha\beta} - 1/2Rg^{\alpha\beta})$ is unbounded on $\gamma([0, v])$.

For a RW spacetime, the invariant $(R_{\alpha\beta} - 1/2Rg_{\alpha\beta})(R^{\alpha\beta} - 1/2Rg^{\alpha\beta})$ is found to be

$$\frac{12k}{a^4} + \frac{24\dot{a}^2k}{a^4} + \frac{12\dot{a}^4}{a^4} + \frac{12\ddot{a}}{a^3} + \frac{12\dot{a}^2\ddot{a}}{a^3} + \frac{12\ddot{a}^2}{a^2}, \quad (4.4.5)$$

and since $a \rightarrow a(t_s)$, $H(t) \rightarrow H_s$, $p(t) \rightarrow \infty$, $\ddot{a}/a \rightarrow -\infty$ as $t \rightarrow t_s$, we see that $(R_{\alpha\beta} - 1/2Rg_{\alpha\beta})(R^{\alpha\beta} - 1/2Rg^{\alpha\beta})$ is unbounded at t_s . Hence, we find that such a spacetime is geodesically incomplete. Note that this holds true irrespectively of the exact form of the scale factor, we only use the qualitative properties that define a sudden singularity (Definition 3.4.1).

Further, the relative asymptotic strength of a sudden singularity is immediately implied by the definition of this type of singularity and it is necessarily of the form $(H \sim |D| \sim a) \ll |E|$.

4.4.2 Fluids with p and μ connected

Although Barrow in [56] kept the pressure and the density unconnected in his example of a sudden singularity, Nojiri et al. in [52] focused on finding specific equations of state appropriate for the emergence of such a structure in a flat universe. An example of such an equation of state is given in [52],

$$p + \mu = -C(\mu_0 - \mu)^{-\gamma}, \quad (4.4.6)$$

where γ is a positive constant. We are interested below in the asymptotic behaviour of the resulting solution of the cosmological equations in the limit of $\mu \rightarrow \mu_0$. Integrating the continuity equation, we find that the scale factor, given by

$$a = a_0 \exp\left(-\frac{(\mu_0 - \mu)^{\gamma+1}}{3C(\gamma + 1)}\right) \quad (4.4.7)$$

(a_0 is an integration constant), remains finite as $\mu \rightarrow \mu_0$. From the Friedmann equation $3H^2 = \mu$, it follows that H and therefore $|D|$ must also remain finite. However, in the limit $\mu \rightarrow \mu_0$, the pressure, and hence $|E|$ blow up. Solving the Friedmann equation for H , substituting in the continuity equation $\dot{\mu} = 3HC(\mu_0 - \mu)^{-\gamma}$, and integrating we find

$$t = t_0 + \int \frac{d\mu}{\sqrt{3\mu}C(\mu_0 - \mu)^{-\gamma}}, \quad (4.4.8)$$

which remains finite (since the integrand decreases to zero) even when $\mu \rightarrow \mu_0$ (recall that $\gamma > 0$). This fact leads to the conclusion that singular behaviour is met within a finite interval of proper time t and, as the following theorem shows, this type of behaviour continues to be valid also in curved models.

Theorem 4.4.1 *A necessary and sufficient condition for a sudden singularity of type (S_3, N_2, B_3) to occur at t_0 in a universe filled with a fluid with equation of state $p + \mu = -C(\mu_0 - \mu)^{-\gamma}$, $\gamma > 0$, is that $\mu \rightarrow \mu_0$ at t_0 .*

PROOF. Integrating the continuity equation $\dot{\mu} - 3CH(\mu_0 - \mu)^{-\gamma} = 0$, we find

$$a \propto \exp\left\{-\frac{(\mu_0 - \mu)^{\gamma+1}}{3C(\gamma + 1)}\right\}, \quad (4.4.9)$$

which is finite as $t \rightarrow t_0$. Also since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2}, \quad (4.4.10)$$

$$|E|^2 = 3\frac{\ddot{a}^2}{a^2} = \frac{1}{12}(\mu + 3p)^2 = \frac{1}{12}(2\mu + 3C(\mu_0 - \mu)^{-\gamma})^2, \quad (4.4.11)$$

$$|D|^2 = 3\left(H^2 + \frac{k}{a^2}\right)^2 = \frac{\mu^2}{3}, \quad (4.4.12)$$

we see that as $t \rightarrow t_0$, H and $|D|$ remain finite whereas $|E|$ diverges, leading to a sudden singularity. The converse is immediate. ■

4.4.3 Milder-than-sudden singularities

A sudden singularity is a very mild type of singularity in the sense that the only diverging physical quantity that characterises it is the $|E|$ component of the Bel-Robinson energy. The only other milder type of singularity that could occur would be one with a bounded Bel-Robinson energy as this is described by the B_4 case in our classification scheme. This can indeed happen. Consider a flat universe filled with a fluid with equation of state (cf. [52])

$$p + \mu = -\frac{AB\mu^{2\beta-1}}{A\mu^{\beta-1} + B}, \quad 0 < \beta < 1/2. \quad (4.4.13)$$

Such a fluid, admits the exact solution $a = a_0 e^{\tau^{8/3}}$ when $\beta = 1/5$. Then $H = (8/3)\tau^{5/3}$, $\dot{H} = (40/9)\tau^{2/3}$ and $\ddot{H} = (80/27)\tau^{-1/3}$, and thus as $\tau \rightarrow 0$, a , \dot{a} , \ddot{a} , H , \dot{H} all remain finite whereas \ddot{H} becomes divergent. We can see that in this model the Bel Robinson energy at the initial time, $\mathcal{B}(0)$, is finite whereas *its time derivative* is given by

$$\dot{\mathcal{B}}(\tau) = 3\left[2\frac{\ddot{a}}{a}(\ddot{H} + 2H\dot{H}) + 4\left(\frac{k}{a^2} + H^2\right)\left(-\frac{kH}{a^2} + H\dot{H}\right)\right], \quad (4.4.14)$$

hence $\dot{\mathcal{B}}(\tau) \rightarrow \infty$ at $\tau \rightarrow 0$. Since the derivative of the Bel-Robinson energy diverges we may interpret this singularity geometrically as one *in the velocities of the Bianchi (frame) field*. At t_s , the Bianchi field encounters a cusp and its velocity diverges there [70]. We believe this to be the mildest type of singularity known in the literature to date.

Another example of a very mild singularity is given by Borde *et al* in [89]. These

authors define an averaged-out Hubble function, H_{av} (given by Eq. (4.4.18) below), along a null or timelike geodesic and showed that every past-directed null or timelike geodesic satisfying $H_{av} > 0$ throughout the past is not past complete. To see this consider first a null geodesic with affine parameter λ . Then using Eq. (2.2.5) and substituting the Christoffel symbols (see footnote (1) in p. 14), we find that the geodesic equation is given by

$$\frac{d^2 t}{d\lambda^2} + \frac{\dot{a}}{a} \left(a^2 \left(\frac{dr}{d\lambda} \right)^2 + f^2 a^2 \left(\frac{d\theta}{d\lambda} \right)^2 + a^2 f^2 \sin^2 \theta \left(\frac{d\phi}{d\lambda} \right)^2 \right) = 0. \quad (4.4.15)$$

Using the fact that the geodesic is null ($ds^2 = 0$) we have that

$$\frac{d^2 t}{d\lambda^2} + \frac{\dot{a}}{a} \left(\frac{dt}{d\lambda} \right)^2 = 0 \quad (4.4.16)$$

and by integration we find that

$$d\lambda \propto a(t) dt. \quad (4.4.17)$$

Setting $d\lambda = (a(t)/a(t_s)) dt$, the affine parameter can be normalised so that $d\lambda/dt = 1$ for $t = t_s$. The averaged-out Hubble function is defined by

$$H_{av} = \frac{1}{\lambda(t_s) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_s)} H(\lambda) d\lambda. \quad (4.4.18)$$

Now by considering those null geodesics that satisfy $H_{av} > 0$, we find

$$0 < H_{av} = \frac{1}{\lambda(t_s) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_s)} H(\lambda) d\lambda = \frac{1}{\lambda(t_s) - \lambda(t_i)} \int_{a(t_i)}^{a(t_s)} \frac{da}{a(t_s)} < \frac{1}{\lambda(t_s) - \lambda(t_i)}. \quad (4.4.19)$$

This shows that the affine parameter of *every* past-directed null geodesic having $H_{av} > 0$ must take values only in a finite interval, which implies geodesic incompleteness. A similar proof is obtained for the case of a timelike geodesic. Hence, all causal past-directed geodesics obeying the condition $H_{av} > 0$ are past incomplete.

Note that for null geodesics, Eq. (4.4.19) implies that H is finite but *only* in a finite time interval (an analogous conclusion holds for a timelike geodesic) and this is no other

than the S_3 (necessary) condition for a singularity of Theorem 3.2.1. We therefore conclude that under the assumptions of [89], privileged observers cannot exist in such expanding universes for an infinite proper time because had they existed Theorem 2.2.2 would imply that spacetime ought to be timelike and null geodesically complete.

4.5 Sudden-rip singularities

A sudden-rip singularity arises once we allow not only $|E|$ but also $|D|$ to diverge at a finite time, while the scale factor remains finite there. This of course implies that H , p and μ become also divergent. This possibility was explored in some examples in [52]. An appropriate equation of state to illustrate this type of asymptotic behaviour is to take [52]

$$p + \mu = -B\mu^\beta, \quad \beta > 1. \quad (4.5.1)$$

Integration of the continuity equation $\dot{\mu} - 3BH\mu^\beta = 0$ then gives directly [52]

$$a = a_0 \exp\left(\frac{\mu^{1-\beta}}{3(1-\beta)B}\right), \quad (4.5.2)$$

which is finite everywhere since we have chosen $\beta > 1$. We can find the relation between the proper time and the density by first substituting Eq. (4.5.1) in the continuity equation and then using the Friedmann equation $3H^2 = \mu$. We obtain

$$dt = \int \frac{d\mu}{\sqrt{3\mu}B\mu^\beta}, \quad (4.5.3)$$

or equivalently [52]

$$t \sim t_0 + \frac{2}{\sqrt{3}B} \frac{\mu^{-\beta+1/2}}{1-2\beta}. \quad (4.5.4)$$

Therefore, when $\mu \rightarrow \infty$, $t \rightarrow t_0$ and so $|p|$, H , $|D|$ and $|E|$ all diverge within the finite time t_0 , meaning that this is an (S_2, N_2, B_1) singularity. The asymptotic strength of this singularity is found to be: $a \ll H \ll |D| \ll |E|$.

This behavior is a special case of the following general result which holds also in curved models.

Theorem 4.5.1 *A necessary and sufficient condition for a sudden-rip singularity (of type (S_2, N_2, B_1)) to occur at t_s in an isotropic universe filled with a fluid with equation of state $p + \mu = -B\mu^\beta$, $\beta > 1$, is that $\mu \rightarrow \infty$ as $t \rightarrow t_s$.*

PROOF. From the continuity equation we have

$$a = a_0 \exp\left(\frac{\mu^{1-\beta}}{3B(1-\beta)}\right), \quad (4.5.5)$$

and so $a \rightarrow a_0$ as $t \rightarrow t_s$. Since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2}, \quad |E|^2 = \frac{1}{12}(2\mu + 3B\mu^\beta)^2, \quad |D|^2 = \frac{\mu^2}{3}, \quad (4.5.6)$$

we see that as $t \rightarrow t_s$, H , $|E|$ and $|D|$ diverge leading precisely to a sudden-rip singularity. The converse is obvious. ■

4.6 Discussion

We have illustrated in this chapter that our methods of classifying singularities through the behaviour of the Hubble parameter the scale factor and the Bel-Robinson energy provide a rigorous basis for a useful description of the dynamics of a variety of cosmological models that exhibit finite-time singularities. Even though these model universes come from completely different motivations yet they reveal their tendencies to have finite-time singularities in the sense of geodesic incompleteness for primarily one reason, the non-integrability of H . Finite-time singularities such as collapse, big-rip or sudden singularities occur because H fails to be integrable for an infinite time interval. In such universes, privileged observers cannot exist for an infinite proper time. We know from Theorem 2.2.2 that if such observers existed for an infinite proper time, then the universe would be timelike and null geodesi-

cally complete. Indeed, this is the case in various families of complete models ranging from dark energy, to inflationary to braneworlds that we have already studied.

Naturally, the fact that the non-integrability condition for H can be sustained in different models stems from the various ways that these models are constructed and the specific features they share. This non-integrability may be supported either by the presence of phantom-dark energy driving the observed cosmic acceleration and leading to a divergence of both $|E|$ and $|D|$, or by a fluid with only a diverging $|E|$ as in [56], or in the inflationary character of the specific model as in [89], etc. All these different particularities are acknowledged and brought to light with the use of the Bel-Robinson energy. We have shown that with the combination of our classifying methods we have succeeded to obtain a complete picture of the reasons underlying a wide variety of possible dynamical behaviours of the universe on approach to a spacetime singularity.

Chapter 5

Braneworld singularities

5.1 Introduction

In previous chapters we studied singularities that arise within the framework of a four-dimensional RW-spacetime that models the whole entity of the observed universe. In this chapter we study possible singularities that develop when we consider the different case in which the observed universe is modeled by a four-dimensional hypersurface situated at a fixed position of an extra spatial dimension. The whole spacetime is five dimensional: there are four dimensions of space out of which only three are spanned by the hypersurface and one dimension of time; such a hypersurface is called a *3-brane* while the full higher dimensional space is called the *bulk*.

Although a brane can trap particles and forces making it impossible for them to escape, it *does* interact with the matter and forces in the bulk. This happens because gravity naturally extends over all dimensions of space and time and it can therefore influence the fields that are constrained to live within the boundaries of the brane [139]. Other matter fields in the bulk can also interact with the fields on the brane, and the strength of this interaction is controlled by a coupling function that is model-dependent [139].

Branes were discovered within the framework of string theory as locations where open strings are constrained to end. Their importance was reinforced when it was realised that

they can play a catalytic role in proving duality between the different versions of string theory. Driven by these new discoveries, a great deal of recent research focuses on building model universes that include branes (see Refs. [140]-[197] for a partial list).

All these different model universes that flourish in the recent literature are termed *braneworlds* and in many cases provided satisfactory explanations to some of the most challenging unresolved issues of cosmology and particle physics such as the hierarchy problem and the cosmological constant problem. Below, we refer briefly to the ideas behind three of the most recognised braneworld cosmological scenarios and to the implications they offer for the singularity problem. The first scenario is the one introduced by Nima Arkani-Hamed, Savas Dimopoulos and Gia Dvali in [198] dubbed as the ADD scenario and the two scenarios that follow are those introduced by Lisa Randall and Raman Sundrum in [199] and [200] known as RS¹ and RS².

According to the ADD scenario all standard particles and forces except gravity are confined on a brane while the extra dimension is as big as a tenth of a millimeter; this is surprising since until then it was believed that extra dimensions were significantly smaller. In this scenario the hierarchy problem, that poses the question of why gravity at small distances appears to be so much weaker than the other forces, *is* solved: the effect of gravity on particles could fade away because of the large extra dimension so that gravity would appear much weaker than the brane-restricted electromagnetic strong and weak forces whose strength remains unaffected exactly because they cannot experience the extra dimension.

A different approach to the hierarchy problem is provided in [199] and [200]. In the RS¹ scenario the extra dimension is considered to be small in size and bounded between two branes called the Gravitybrane and the Weakbrane. In this model the authors make essential use of a warped geometry where the *warp factor* that defines how space, time, energies and masses change with distance (measured along the fifth extra dimension) decreases exponentially along the fifth dimension. A consequence of this configuration is that the energy of the bulk and the branes warps the spacetime in such a way that gravity is felt

strong on the Gravitybrane and weak on the Weakbrane which accommodates all standard particles and forces. In contrast, in the RS² scenario where there is only one brane, the extra dimension is infinite in size but it can nonetheless remain invisible provided that the spacetime is suitably warped.

In this chapter we use the concept of a warped geometry in our braneworld models but we will not focus so much on the exploration of the phenomenological implications that these models offer, rather on the interesting features of finite-distance singularities that they exhibit as well as on very basic questions (to which we refer below) that the development of such singularities erases.

In particular, we will focus on the analysis of a new version of a braneworld model that was originally proposed by Arkani-Hamed *et al.* in [74]. In the original version [74] the model consists of a flat three-brane embedded in a five dimensional bulk space that is filled with a scalar field minimally coupled to the bulk and conformally coupled to the fields on the brane. The metric describing the five dimensional spacetime is a warped product as in the RS scenarios. A basic feature of this model is the development of a singularity at a finite distance from the brane around which the warp factor, its derivative and the energy density of the scalar field all diverge. This singularity was used in [74] in order to provide an explanation to the cosmological constant problem while preserving four dimensional flatness. In particular, the authors in [74] argue that this singularity on the one hand successfully cuts off the size of the extra dimension giving rise to the observed four dimensional flatness, and on the other hand it acts as a reservoir through which all of the vacuum energy is emptied so that its value is brought to agreement with the observed one.

Our interest in this particular braneworld model springs precisely from the fact that it exhibits a finite-distance singularity which bares resemblance to the finite-time singularities that are the core of our research. Our quest is to provide answers to the following questions that naturally arise: How general is the behaviour found in [74]? Is the finite-distance singularity the only possible outcome when we pass on to a curved instead of a flat brane?

How strongly are the dynamical features of the model affected by a different form of bulk matter such as a perfect fluid? Is the type of singularity found in [74] the only possible one in these new braneworld models, and if not, what other types of finite-distance singularities can arise?

In this chapter we show that in a curved brane the finite-distance singularity found in [74] shifts to an infinite distance away from the brane, while with a perfect fluid in the bulk new types of finite-distance singularities emerge. On approach to these singularities the warp factor can do only two things: It will either tend to zero (collapse-brane I or II), or diverge (big rip-brane singularities¹) while the energy density of the bulk-matter components usually diverges. Our analysis implies that the effect of an infinite energy leak to the extra dimension is indeed generic. For a qualitative analysis of such models the method of asymptotic splittings, expounded in Appendix E, is a most useful tool.

The plan of this chapter is as follows: In Section 5.2, we review the basic aspects of the embedding of a three-brane in a five dimensional bulk space. In Section 5.3, we give the field equations that describe the dynamics of a three-brane embedded in a five dimensional bulk with a scalar field; in Subsection 5.3.1, we analyse the possible asymptotic behaviours of that model using the method of asymptotic splittings. In Section 5.4, we explore the consequences of altering the matter in the bulk. In particular, the case where the bulk is filled with a perfect fluid has a dramatic effect on the global dynamics. We no longer encounter just the singularities of the type found in [74] but now we have dynamics characterised by the finite-distance singularities referred to in the previous paragraph. In Section 5.6, we conclude and discuss our results.

5.2 Brane geometry

Consider a four-dimensional hypersurface Σ that is situated at the arbitrary but fixed position Y_* where Y is the coordinate along the fifth dimension of the embedding space. The brane has negligible thickness compared to the extra dimension (and also the other

¹See Def. 5.4.1

dimensions) extending over the higher dimensional bulk space and can be viewed as a thin surface (surface layer). The brane therefore naturally divides the bulk in two regions V^+ and V^- each containing one of the boundary surfaces Σ^+ and Σ^- that confine the brane. The two regions V^+ and V^- are expected to look identical from Σ^+ and Σ^- , the only difference being that their normal vector should be reversed; in other words our braneworld admits a $Y \rightarrow -Y$ symmetry. The extrinsic curvatures of the two embeddings, K_{ab}^+ and K_{ab}^- , therefore satisfy

$$K_{ab}^+ = -K_{ab}^-, \quad \text{everywhere on } \Sigma. \quad (5.2.1)$$

Using the Gauss-Codazzi formalism, analysed in Appendix B.2, we can write the Ricci 5-tensor in a form analogous to (B.2.20) (note that $\epsilon(n) = 1$), i.e.,

$$R_{ab} = {}^{(4)}R_{ab} - K_{ab}K + 2K_{an}K_b^n - \frac{\partial K_{ab}}{\partial Y}, \quad (5.2.2)$$

or

$$R_{ab} = -\frac{\partial K_{ab}}{\partial Y} + Z_{ab}, \quad (5.2.3)$$

where

$$Z_{ab} = {}^{(4)}R_{ab} - K_{ab}K + 2K_{an}K_b^n. \quad (5.2.4)$$

Tracing the Einstein equations

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa_5^2 T_{ab}, \quad (5.2.5)$$

where $\kappa_5^2 = 8\pi G_5 = M_5^{-3}$ with G_5 being the five dimensional Newton's constant and M_5 the five dimensional Planck mass, we find that the scalar curvature is given by

$$R = -\frac{2}{3}\kappa_5^2 T. \quad (5.2.6)$$

Substituting back in (5.2.5) leads to the following equivalent form of the Einstein equations:

$$R_{ab} = \kappa_5^2 \left(T_{ab} - \frac{1}{3} g_{ab} T \right). \quad (5.2.7)$$

Inserting Eq. (5.2.3) in Eq. (5.2.7) and integrating over the layer defined by the thickness of the brane we obtain [201], [202]

$$-K_{ab}^+ + K_{ab}^- + \int_{Y_*-\epsilon}^{Y_*+\epsilon} Z_{ab} dY = \kappa_5^2 \left(T_{ab} - \frac{1}{3} g_{ab} T \right). \quad (5.2.8)$$

In the passage to the limit $\epsilon \rightarrow 0$ and for fixed $-K_{ab}^+$, K_{ab}^- we naturally expect that K_{ab} remains bounded inside the layer and hence the integral of Z_{ab} over the layer should vanish [201], [202]. Taking thus the limit of the above equation for $\epsilon \rightarrow 0$ and using Eq. (5.2.1), we find the following *junction condition* [201], [202] (see also [203] pp. 551-556)

$$[K_{ab}] \equiv K_{ab}^+ - K_{ab}^- = 2K_{ab}^+ = -\kappa_5^2 \left(S_{ab} - \frac{1}{3} g_{ab} S \right), \quad (5.2.9)$$

that relates the jump of the extrinsic curvature $[K_{ab}]$ across the brane to the surface-energy momentum tensor S_{ab} (this follows from the discussion below) defined by

$$S_{ab} \equiv \lim_{\epsilon \rightarrow 0} \int_{Y_*-\epsilon}^{Y_*+\epsilon} T_{ab} dY, \quad (5.2.10)$$

with $S = g^{ab} S_{ab}$ being the trace of S_{ab} . It can be proved (cf. [201], [202], and [203] pp. 551-556) that integration of the rest of the components of the Einstein equations gives S_{55} and S_{5a} with $a = 1, 2, 3, 4$ both zero and hence, we conclude that there is no flow of energy from the fields restricted on Σ into the bulk. It is thus implied that the quantity S_{ab} acts as a delta function source of matter existing only on Σ and vanishing off Σ . This last property suggests that S_{ab} plays the role of a surface-energy momentum tensor. Combining this with Eq. (5.2.9), it follows that the extrinsic curvature of the brane is completely determined by the matter confined inside the brane [204].

5.3 Asymptotics of scalar field-brane configuration

Consider the braneworld model consisting of the three-brane configuration analysed in the previous Section and filled with a scalar field minimally coupled to the five-dimensional bulk and conformally coupled to any fields restricted on the brane. The total action S_{total} then splits in two parts, namely, the bulk action S_{bulk} and the brane action S_{brane} , i.e.,

$$S_{total} = S_{bulk} + S_{brane}, \quad (5.3.1)$$

with

$$S_{bulk} = \int d^4x dY \sqrt{g_5} \left(\frac{R}{2\kappa_5^2} - \frac{\lambda}{2} (\nabla\phi)^2 \right), \quad (5.3.2)$$

$$S_{brane} = - \int d^4x \sqrt{g_4} f(\phi), \text{ at } Y = Y_*, \quad (5.3.3)$$

where λ is a parameter defining the type of scalar field ϕ , $\kappa_5^2 = M_5^{-3}$, M_5 is the five dimensional Planck mass, and $f(\phi)$ denotes the tension of the brane depending on the scalar field. The tension of the brane can be understood physically as the resistance that one encounters when trying to distort the brane: zero tension means that the brane has no resistance so that any distortion would influence its form dramatically, while infinite tension would deprive the brane's liberty to evolve dynamically [139].

Varying the total action (5.3.1) with respect to g^{AB} , we find the five-dimensional Einstein field equations in the form [74]

$$R_{AB} - \frac{1}{2}g_{AB}R = \lambda\kappa_5^2 \left(\nabla_A\phi\nabla_B\phi - \frac{1}{2}g_{AB}(\nabla\phi)^2 \right) - \frac{2\kappa_5^2}{\sqrt{g_5}} \frac{\delta S_{brane}}{\delta g^{\alpha\beta}} \delta_\alpha^A \delta_\beta^B \delta(Y),^2 \quad (5.3.4)$$

while the scalar field equation is obtained by variation of the action (5.3.1) with respect to ϕ [74]³

²Note here that by working in a similar way as in Appendix D, we may interpret the energy-momentum tensor of the scalar field as that of a perfect fluid in the bulk having density $\rho = \rho(Y)$ and pressure $P = P(Y)$, i.e., $T_{AB} = (\rho + P)u_A u_B - P g_{AB}$, where $u_A = (0, 0, 0, 0, 1)$, so that $P = 1/2(\nabla\phi)^2 = \rho$.

³The left-hand side of this equation is the analogous left-hand side of the equation of motion (D.0.7)

$$\lambda \square_5 \phi = -\frac{1}{\sqrt{g_4}} \frac{\delta S_{brane}}{\delta \phi} \delta(Y), \quad (5.3.5)$$

where $A, B = 1, 2, 3, 4, 5$ and $\alpha, \beta = 1, 2, 3, 4$ while $\delta(Y) = 1$ at $Y = Y_*$ and vanishing everywhere else.

We assume a bulk metric of the form

$$ds^2 = a^2(Y) d\tilde{s}^2 + dY^2, \quad (5.3.6)$$

where $d\tilde{s}^2$ is the four dimensional flat, de Sitter or anti de Sitter metric, i.e.,

$$d\tilde{s}^2 = \begin{cases} -dt^2 + dr^2 + r^2 d\sigma^2 \\ -dt^2 + \frac{\cosh^2(Ht)}{H^2} (dr^2 + \sin^2 r d\sigma^2) \\ -dt^2 + \frac{\cos^2(Ht)}{H^2} (dr^2 + \sinh^2 r d\sigma^2), \end{cases} \quad (5.3.7)$$

where $d\sigma^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. The field equations then take the form

$$\frac{a'^2}{a^2} = \frac{\lambda \kappa_5^2 \phi'^2}{12} + \frac{kH^2}{a^2} \quad (5.3.8)$$

$$\frac{a''}{a} = -\frac{\lambda \kappa_5^2 \phi'^2}{4} \quad (5.3.9)$$

$$\phi'' + 4 \frac{a'}{a} \phi' = 0, \quad (5.3.10)$$

where $k = 0, \pm 1$ and H^{-1} is the de Sitter curvature radius and the prime denotes differentiation with respect to Y . The variables to be determined are a , a' and ϕ . These three equations are not independent since Eq. (5.3.9) was derived after substitution of Eq. (5.3.8) in the field equation $G_{\alpha\alpha} = \kappa_5^2 T_{\alpha\alpha}$, $\alpha = 1, 2, 3, 4$,

$$\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{kH^2}{a^2} = -\lambda \kappa_5^2 \frac{\phi'^2}{6}. \quad (5.3.11)$$

In our analysis we use the independent equations (5.3.9) and (5.3.10) to determine the

of a scalar field in a four-dimensional spacetime with $V(\phi) = 0$ (see appendix D).

unknown variables a , a' and ϕ' and the Eq. (5.3.8) will then play the role of a constraint equation for our system.

The extrinsic curvature of the three-brane is calculated to be

$$K_{\alpha\beta} = \frac{1}{2} \frac{\partial}{\partial Y} g_{\alpha\beta} = a'(Y)a(Y), \quad (5.3.12)$$

while the surface energy-momentum tensor (defined only on the brane and vanishing off the brane) is now given by

$$S_{\alpha\beta} = -\frac{2\kappa_5^2}{\sqrt{g_5}} \frac{\delta S_{brane}}{\delta g^{\alpha\beta}} \delta_\alpha^A \delta_\beta^B \delta(Y) = -a^2(Y_*)f(\phi(Y_*)). \quad (5.3.13)$$

Solving Eq. (5.3.4) on the brane is equivalent to substituting Eqs. (5.3.12) and (5.3.13) in condition (5.2.9) which leads to the following relation to be satisfied by $a'(Y_*)$:

$$a'(Y_*) = -\frac{\kappa_5^2}{6} f(\phi(Y_*))a(Y_*). \quad (5.3.14)$$

On the other hand, Eq. (5.3.5) gives directly

$$\frac{\lambda}{\sqrt{g_5}} \frac{\partial}{\partial Y} (\sqrt{g_5} \phi') = \frac{1}{\sqrt{g_4}} \int dx^4 \sqrt{g_4} f'(\phi) \delta(Y). \quad (5.3.15)$$

Integrating over the interval $[Y_* - \epsilon, Y_* + \epsilon]$ and taking the limit as $\epsilon \rightarrow 0$, we finally find

$$\phi'(Y_*) = \frac{f'(\phi(Y_*))}{2\lambda}, \quad (5.3.16)$$

where we have used the $Y \rightarrow -Y$ symmetry, i.e., $\phi'^+ = \phi'(Y_* + \epsilon) = -\phi'(Y_* - \epsilon) = -\phi'^-$.

Our purpose now is to find all possible asymptotic behaviours around the assumed position of a singularity, denoted by Y_s , emerging from general or particular solutions of the system of equations (5.3.8)-(5.3.10) describing our braneworld cosmology. As we mentioned in the Introduction a most useful tool for this analysis is the method of asymptotic splittings

([76], see also Appendix E). According to this method, we start by setting

$$x = a, \quad y = a', \quad z = \phi', \quad (5.3.17)$$

where the differentiation is considered with respect to $\Upsilon = Y - Y_s$. The field equations (5.3.9) and (5.3.10) become the following system of ordinary differential equations:

$$x' = y \quad (5.3.18)$$

$$y' = -\lambda A z^2 x \quad (5.3.19)$$

$$z' = -4y \frac{z}{x}, \quad (5.3.20)$$

where $A = \kappa_5^2/4$. Hence, we have the vector field

$$\mathbf{f} = \left(y, -\lambda A z^2 x, -4y \frac{z}{x} \right)^\top. \quad (5.3.21)$$

Equation (5.3.8) does not include any terms containing derivatives with respect to Υ ; it is to be regarded as a constraint equation for the above system (5.3.18)-(5.3.20). In terms of the new variables the constraint is written in the form

$$\frac{y^2}{x^2} = \frac{A\lambda}{3} z^2 + \frac{kH^2}{x^2}. \quad (5.3.22)$$

Substituting the forms

$$(x, y, z) = (\alpha \Upsilon^p, \beta \Upsilon^q, \delta \Upsilon^r),$$

where

$$(p, q, r) \in \mathbb{Q}^3 \quad \text{and} \quad (\alpha, \beta, \delta) \in \mathbb{C}^3 \setminus \{\mathbf{0}\}, \quad (5.3.23)$$

in the dynamical system (5.3.18)-(5.3.20), we seek to determine the possible *dominant balances* in the neighborhood of the singularity, that is pairs of the form

$$\mathcal{B} = \{\mathbf{a}, \mathbf{p}\}, \quad \text{where} \quad \mathbf{a} = (\alpha, \beta, \delta), \quad \mathbf{p} = (p, q, r). \quad (5.3.24)$$

For our system (5.3.18)-(5.3.20) we find[†]:

$$\mathcal{B}_1 = \{(\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda})), (1/4, -3/4, -1)\} \quad (5.3.25)$$

$$\mathcal{B}_2 = \{(\alpha, \alpha, 0), (1, 0, -1)\} \quad (5.3.26)$$

Since (5.3.18)-(5.3.20) is a weight-homogeneous system, the scale invariant solutions given by the above balances are exact solutions of the system. The balance \mathcal{B}_1 satisfies the constraint Eq. (5.3.22) only for $k = 0$ corresponding thus to a general (as we will see in the next paragraphs) solution that allows *only* for a flat spatial brane geometry. In contrast, \mathcal{B}_2 corresponds to a particular solution for a *curved brane* since it satisfies Eq. (5.3.22) for $k \neq 0$ and $\alpha^2 = kH^2$ (here we have to sacrifice one arbitrary constant by setting it equal to kH^2).

Let us now focus on each of the two possible balances separately and build series expansions in the neighborhood of the singularity. We start by calculating the Kowalevskaya exponents, eigenvalues of the matrix $\mathcal{K} = D\mathbf{f}(\mathbf{a}) - \text{diag}(\mathbf{p})$, where $D\mathbf{f}(\mathbf{a})$ is the Jacobian matrix of f ,

$$D\mathbf{f}(x, y, z) = \begin{pmatrix} 0 & 1 & 0 \\ -A\lambda z^2 & 0 & -2A\lambda xz \\ \frac{4yz}{x^2} & -\frac{4z}{x} & -\frac{4y}{x} \end{pmatrix}, \quad (5.3.27)$$

evaluated on \mathbf{a} . For the first balance we have that $\mathbf{a} = (\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}))$, and $\mathbf{p} = (1/4, -3/4, -1)$, thus

$$\mathcal{K} = \begin{pmatrix} \frac{1}{4} & 1 & 0 \\ -\frac{3}{16} & \frac{3}{4} & -\frac{\sqrt{3A\lambda}\alpha}{2} \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & 0 \end{pmatrix}. \quad (5.3.28)$$

^{3†} We have ignored here a third balance $\mathcal{B}_3 = \{(\alpha, 0, 0), (0, -1, -1)\}$ since it does not give the necessary -1 as one of the Kowalevskaya exponents (see Appendix E for more details).

The \mathcal{K} -exponents are then given by

$$\text{spec}(\mathcal{K}) = \{-1, 0, 3/2\}. \quad (5.3.29)$$

As it follows from the analysis in Appendix E, these exponents correspond to the indices of the series coefficients where arbitrary constants first appear. The -1 exponent signals the arbitrary position of the singularity, Y_s . This first balance has two non-negative rational eigenvalues which means that it describes a general solution, i.e., a solution having the full number of arbitrary constants. In order to construct an asymptotic expansion of this solution, we substitute in the system (5.3.18)-(5.3.20) the series expansions

$$\mathbf{x} = \Upsilon^{\mathbf{P}}(\mathbf{a} + \sum_{j=1}^{\infty} \mathbf{c}_j \Upsilon^{j/s}),$$

where $\mathbf{x} = (x, y, z)$, $\mathbf{c}_j = (c_{j1}, c_{j2}, c_{j3})$, and s is the least common multiple of the denominators of the positive eigenvalues; here $s = 2$, and the corresponding series expansions are given by,

$$x = \sum_{j=0}^{\infty} c_{j1} \Upsilon^{j/2+1/4}, \quad y = \sum_{j=0}^{\infty} c_{j2} \Upsilon^{j/2-3/4}, \quad z = \sum_{j=0}^{\infty} c_{j3} \Upsilon^{j/2-1}. \quad (5.3.30)$$

Following this, we arrive at the asymptotic solution around the singularity in the form

$$x = \alpha \Upsilon^{1/4} + \frac{4}{7} c_{32} \Upsilon^{7/4} + \dots \quad (5.3.31)$$

$$y = \frac{\alpha}{4} \Upsilon^{-3/4} + c_{32} \Upsilon^{3/4} + \dots \quad (5.3.32)$$

$$z = \frac{\sqrt{3}}{4\sqrt{A\lambda}} \Upsilon^{-1} - \frac{4\sqrt{3}}{7\alpha\sqrt{A\lambda}} c_{32} \Upsilon^{1/2} + \dots \quad (5.3.33)$$

The last step is to check if for each j satisfying $j/2 = \rho$ with ρ a positive eigenvalue, the corresponding eigenvector v of the \mathcal{K} matrix is such that the *compatibility conditions* hold, namely,

$$v^{\top} \cdot P_j = 0, \quad (5.3.34)$$

where P_j are polynomials in $\mathbf{c}_i, \dots, \mathbf{c}_{j-1}$ given by

$$\mathcal{K}\mathbf{c}_j - (j/s)\mathbf{c}_j = P_j. \quad (5.3.35)$$

Here the corresponding relation $j/2 = 3/2$, is valid only for $j = 3$ and the associated eigenvector is

$$v = \left(-\frac{\alpha\sqrt{A\lambda}}{\sqrt{3}}, -\frac{7\alpha\sqrt{A\lambda}}{4\sqrt{3}}, 1 \right). \quad (5.3.36)$$

The compatibility condition, $v \cdot (\mathcal{K} - (3/2)\mathcal{I}_3)\mathbf{c}_3 = 0$, therefore indeed holds since,

$$(\mathcal{K} - (3/2)\mathcal{I}_3)\mathbf{c}_3 = c_{32} \begin{pmatrix} -\frac{7}{4} & 1 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{7} \\ 1 \\ -\frac{4\sqrt{3}}{7\alpha\sqrt{A\lambda}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.3.37)$$

This shows that a representation of the solution asymptotically with a Puiseux series as this is given by Eqs. (5.3.31)-(5.3.33) is valid. We thus conclude that near the singularity at finite distance Y_s from the brane the asymptotic forms of the variables are

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty. \quad (5.3.38)$$

This is exactly the asymptotic behaviour of the solution found previously by Arkani-Hammed *et al* in [74].

However, the previous behaviour is not the only possible one; the analysis below shows that a very different outcome results by the asymptotic analysis of the second balance. The \mathcal{K} -matrix of the second balance is

$$\mathcal{K} = D\mathbf{f}((\alpha, \alpha, 0)) - \text{diag}(1, 0, -1) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad (5.3.39)$$

with eigenvalues

$$\text{spec}(\mathcal{K}) = \{-1, 0, -3\}. \quad (5.3.40)$$

For the balance \mathcal{B}_2 we find two distinct negative integer \mathcal{K} -exponents and hence we expect to find an infinite negative expansion of a *particular* solution (recall that we had to sacrifice one arbitrary constant) around the presumed singularity at Y_s with the negative \mathcal{K} -exponents signaling the positions where the arbitrary constants first appear [205]. We therefore expand the variables in series with descending powers of Υ in order to meet the two arbitrary constants occurring for $j = -1$ and $j = -3$, i.e.,

$$x = \sum_{j=0}^{-\infty} c_{j1} \Upsilon^{j+1}, \quad y = \sum_{j=0}^{-\infty} c_{j2} \Upsilon^j, \quad z = \sum_{j=0}^{-\infty} c_{j3} \Upsilon^{j-1}. \quad (5.3.41)$$

Substituting these series expansions back in the system (5.3.18)-(5.3.20), we find the following asymptotic behaviour

$$x = \alpha \Upsilon + c_{-11} + \dots \quad (5.3.42)$$

$$y = \alpha + \dots \quad (5.3.43)$$

$$z = c_{-33} \Upsilon^{-4} + \dots \quad (5.3.44)$$

We now check the compatibility condition for $j = -3$ (the associated eigenvector here is $v = (0, 0, 1)$). We find that

$$P_{-3} = (\mathcal{K} + 3\mathcal{I}_3)\mathbf{c}_{-3} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (5.3.45)$$

so that the compatibility condition is indeed satisfied. The expansions given by Eqs. (5.3.42)-(5.3.44) are therefore valid and we can say that as $\Upsilon \rightarrow 0$, or equivalently as $S = 1/\Upsilon \rightarrow \infty$, we have that

$$a \rightarrow \infty, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty. \quad (5.3.46)$$

We thus conclude that there exist two possible outcomes given by Eqs. (5.3.38) and (5.3.46) for these braneworld models, the dynamical behaviours of which strongly depend on the spatial geometry of the brane. For a flat brane the model experiences a finite-distance singularity through which all the vacuum energy decays (since $\phi' \rightarrow \infty$, as $Y \rightarrow Y_s$), whereas for a de Sitter or anti-de Sitter brane the singularity is now located at an infinite distance. Flat solutions correspond to the particular coupling used by Arkani-Hammed *et al* in [74], the only choice to make this possible. This easily follows by using equations (5.3.14) and (5.3.16) and solving the FRW equation (5.3.8) on the brane for kH^2 :

$$kH^2 = \frac{a^2(Y_*)\kappa_5^2}{12} \left(\frac{\kappa_5^2}{3} f^2(\phi(Y_*)) - \frac{f'^2(\phi(Y_*))}{4\lambda} \right).$$

Clearly, k is identically zero if and only if:

$$\frac{f'(\phi)}{f(\phi)} = 2\sqrt{\frac{\lambda}{3}}\kappa_5,$$

or equivalently, if and only if $f(\phi) \propto e^{2\sqrt{\lambda/3}\kappa_5\phi}$ (Arkani-Hammed *et al* in [74] have $\lambda = 3$, and hence the appropriate choice for the brane tension is $f(\phi) \propto e^{2\kappa_5\phi}$).

By working with other couplings we can allow for non-flat, maximally symmetric solutions to exist and avoid having the singularity at a finite distance away from the position of the brane.

In the following Section we replace the scalar field in the bulk with a perfect fluid and explore the outcome of the resulting dynamical system.

5.4 Perfect fluid in the bulk

We consider here the case that the bulk matter takes the form of a perfect fluid with equation of state $P = \gamma\rho$, where P and ρ are functions of Y only. This fluid can be viewed as an entity in its own right but we can also use it to represent an average contribution of the various scalar fields living in the bulk.

The field equations describing the braneworld model in this case are derived by replacing the energy-momentum tensor in the first term of the RHS of Eq. (5.3.4) by that of a perfect fluid, $T_{AB} = (\rho + P)u_A u_B - P g_{AB}$, $u_A = (0, 0, 0, 0, 1)$. They read:

$$\frac{a''}{a} = -\kappa_5^2 \frac{(1 + 2\gamma)}{6} \rho, \quad (5.4.1)$$

$$\rho' + 4(1 + \gamma)H\rho = 0, \quad (5.4.2)$$

$$\frac{a'^2}{a^2} = \frac{\kappa_5^2}{6} A\rho + \frac{kH^2}{a^2}. \quad (5.4.3)$$

Introducing the new variables

$$x = a, \quad y = a', \quad w = \rho, \quad (5.4.4)$$

Eqs. (5.4.1)-(5.4.3) can be written in the form

$$x' = y, \quad (5.4.5)$$

$$y' = -2A \frac{(1 + 2\gamma)}{3} wx, \quad (5.4.6)$$

$$w' = -4(1 + \gamma) \frac{y}{x} w, \quad (5.4.7)$$

with the constraint equation given by

$$\frac{y^2}{x^2} = \frac{2}{3} Aw + \frac{kH^2}{x^2}, \quad A = \kappa_5^2/4. \quad (5.4.8)$$

The system (5.4.5)-(5.4.7) is weight homogeneous with vector field

$$\mathbf{f} = \left(y, -2A \frac{(1+2\gamma)}{3} wx, -4(1+\gamma) \frac{y}{x} w \right)^\top,$$

and by setting $(x, y, w) = (\alpha \Upsilon^p, \beta \Upsilon^q, \delta \Upsilon^s)$, we find the following set of five possible balances:

$${}_\gamma \mathcal{B}_1 = \left\{ \left(\alpha, \alpha p, \frac{3}{2A} p^2 \right), (p, p-1, -2) \right\}, \quad p = \frac{1}{2(\gamma+1)}, \quad \gamma \neq -1/2, -1, \quad (5.4.9)$$

$${}_\gamma \mathcal{B}_2 = \{(\alpha, \alpha, 0), (1, 0, -2)\}, \quad \gamma \neq -1/2, \quad (5.4.10)$$

$${}_{-1/2} \mathcal{B}_3 = \{(\alpha, \alpha, 0), (1, 0, r)\}, \quad (5.4.11)$$

$${}_{-1/2} \mathcal{B}_4 = \{(\alpha, \alpha, \delta), (1, 0, -2)\}, \quad (5.4.12)$$

$${}_{-1/2} \mathcal{B}_5 = \{(\alpha, 0, 0), (0, -1, r)\}, \quad (5.4.13)$$

where ${}_{-1/2} \mathcal{B}_i \equiv_{\gamma=-1/2} \mathcal{B}_i$. These balances are *exact* solutions of the system and they should therefore also satisfy the constraint equation (5.4.8). This fact alters the generality of the solution represented by each balance and determines the type of spatial geometry that it admits: The balances ${}_\gamma \mathcal{B}_1$ and ${}_{-1/2} \mathcal{B}_5$ describe a potentially general solution corresponding to a flat brane, while the balances ${}_\gamma \mathcal{B}_2$ and ${}_{-1/2} \mathcal{B}_3$ describe particular solutions of curved branes since we already have to sacrifice the arbitrary constant α by accepting that $\alpha^2 = kH^2$. The balance ${}_{-1/2} \mathcal{B}_4$ on the other hand describes a particular solution of a curved or flat brane since we have to set $\delta = (3/(2A))(1 - kH^2/\alpha^2)$ in order to satisfy Eq.(5.4.8).

Before we proceed to the analysis of each of these balances in the next Subsections, we find it useful to introduce the following definition of finite-distance singularities:

Definition 5.4.1 *Let a be the warp factor in the metric (5.3.6), a' its derivative and ϱ the energy density of the matter component in the bulk. We call a finite-distance brane-singularity occurring with:*

- i) $a \rightarrow 0$, $a' \rightarrow \infty$ and $\varrho \rightarrow \infty$: a collapse-brane I singularity.*
- ii) $a \rightarrow 0$, $a' \rightarrow c$, where c is an arbitrary constant, and $\varrho \rightarrow \infty$: a collapse-brane II singularity.*

iii) $a \rightarrow \infty$, $a' \rightarrow \infty$ and $\varrho \rightarrow \infty$: a big rip-brane singularity.

Note that with this terminology, the finite-distance singularity we met in the previous Section is a collapse-brane I singularity.

5.4.1 The balance ${}_{\gamma}\mathcal{B}_1$

We begin our analysis by calculating the Kowalevskaya-matrix for the first balance. The Jacobian matrix of \mathbf{f} is given by,

$$D\mathbf{f}(x, y, z) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{2}{3}(1+2\gamma)Aw & 0 & -\frac{2}{3}(1+2\gamma)Ax \\ 4(1+\gamma)\frac{yw}{x^2} & -4(1+\gamma)\frac{w}{x} & -4(1+\gamma)\frac{y}{x} \end{pmatrix} \quad (5.4.14)$$

and

$$\begin{aligned} {}_{\gamma}\mathcal{K}_1 &= D\mathbf{f}\left(\alpha, \alpha p, \frac{3}{2A}p^2\right) - \text{diag}(p, p-1, -2) \\ &= D\mathbf{f}\left(a, \frac{a}{2(1+\gamma)}, \frac{3}{8A(1+\gamma)^2}\right) - \text{diag}\left(\frac{1}{2(1+\gamma)}, -\frac{1+2\gamma}{2(1+\gamma)}, -2\right) \\ &= \begin{pmatrix} -\frac{1}{2(1+\gamma)} & 1 & 0 \\ -\frac{1+2\gamma}{4(1+\gamma)^2} & \frac{1+2\gamma}{2(1+\gamma)} & -\frac{2}{3}(1+2\gamma)A\alpha \\ \frac{3}{4(1+\gamma)^2A\alpha} & -\frac{3}{2(1+\gamma)A\alpha} & 0 \end{pmatrix}. \end{aligned} \quad (5.4.15)$$

The eigenvalues of this matrix are

$$\text{spec}({}_{\gamma}\mathcal{K}_1) = \left\{ -1, 0, \frac{1+2\gamma}{1+\gamma} \right\}. \quad (5.4.16)$$

The last eigenvalue is a function of the γ parameter and it is positive when either $\gamma < -1$ or $\gamma > -1/2$. We are going to focus on each of these two ranges of γ and explore the possible outcome of the system and the character of the finite-distance singularities that occur each time.

Consider first the case with $\gamma < -1$. For purposes of illustration, we take $\gamma = -2$. Then the balance and the ${}_{-2}\mathcal{K}_1$ -exponents read respectively,

$${}_{-2}\mathcal{B}_1 = \{(\alpha, -\alpha/2, 3/(8A)), (-1/2, -3/2, -2)\}, \quad (5.4.17)$$

$$\text{spec}({}_{-2}\mathcal{K}_1) = \{-1, 0, 3\}. \quad (5.4.18)$$

Substituting the value $\gamma = -2$ in the system given by Eqs. (5.4.5)-(5.4.7) and also the forms

$$x = \sum_{j=0}^{\infty} c_{j1} \Upsilon^{j-1/2}, \quad y = \sum_{j=0}^{\infty} c_{j2} \Upsilon^{j-3/2}, \quad z = \sum_{j=0}^{\infty} c_{j3} \Upsilon^{j-2}, \quad (5.4.19)$$

we expect to meet the third arbitrary constant at $j = 3$. Indeed we find,

$$x = \alpha \Upsilon^{-1/2} + \frac{2}{3} A \alpha c_{33} \Upsilon^{5/2} + \dots, \quad (5.4.20)$$

$$y = -\frac{\alpha}{2} \Upsilon^{-3/2} + \frac{5}{3} A \alpha c_{33} \Upsilon^{3/2} + \dots, \quad (5.4.21)$$

$$w = \frac{3}{8A} \Upsilon^{-2} + c_{33} \Upsilon + \dots. \quad (5.4.22)$$

The compatibility condition is trivially satisfied for $j = 3$ since

$$(\mathcal{K} - 3\mathcal{I}_3)\mathbf{c}_3 = \begin{pmatrix} -\frac{5}{2} & 1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 2A\alpha \\ \frac{3}{4A\alpha} & \frac{3}{2A\alpha} & -3 \end{pmatrix} c_{33} \begin{pmatrix} \frac{2}{3}A\alpha \\ \frac{5}{3}A\alpha \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.4.23)$$

The series expansions given by Eqs. (5.4.20)-(5.4.22) are therefore valid asymptotically for

$\Upsilon \rightarrow 0$ so that

$$a \rightarrow \infty, \quad a' \rightarrow -\infty, \quad \rho \rightarrow \infty. \quad (5.4.24)$$

We therefore conclude that the balance ${}_\gamma\mathcal{B}_1$ describes a general solution (corresponding to a flat brane) around a big rip-brane singularity when the perfect fluid satisfies a phantom-like equation of state, i.e., $\gamma < -1$. Note that using the analogy between the warp factor and the scale factor we can say that this singularity bares many similarities to the analogous one arising in the classical theory (see Section 4.3) since it is characterized by all quantities a , ρ and consequently P and H becoming divergent asymptotically.

In order to illustrate the second range of values of γ , $\gamma > -1/2$, we assume that $\gamma = -1/4$. Then

$${}_{-1/4}\mathcal{B}_1 = \{(\alpha, -2\alpha/3, 2/(3A)), (2/3, -1/3, -2)\}, \quad (5.4.25)$$

$$\text{spec}({}_{-1/4}\mathcal{K}_1) = \{-1, 0, 2/3\}. \quad (5.4.26)$$

Substituting in the system (5.4.5)-(5.4.7) the particular value $\gamma = -1/4$ and the forms

$$x = \sum_{j=0}^{\infty} c_{j1} \Upsilon^{j/3+2/3}, \quad y = \sum_{j=0}^{\infty} c_{j2} \Upsilon^{j/3-1/3}, \quad z = \sum_{j=0}^{\infty} c_{j3} \Upsilon^{j/3-2}, \quad (5.4.27)$$

we arrive at the following asymptotic expansions:

$$x = \alpha \Upsilon^{2/3} - \frac{A\alpha}{2} c_{23} \Upsilon^{4/3} + \dots, \quad (5.4.28)$$

$$y = \frac{2}{3} \alpha \Upsilon^{-1/3} - \frac{2}{3} A \alpha c_{23} \Upsilon^{1/3} + \dots, \quad (5.4.29)$$

$$w = \frac{2}{3A} \Upsilon^{-2} + c_{23} \Upsilon^{-4/3} + \dots. \quad (5.4.30)$$

Again, the compatibility condition is bound to be satisfied since in this case

$$(\mathcal{K} - (2/3)\mathcal{I}_3)\mathbf{c}_2 = \begin{pmatrix} -\frac{4}{3} & 1 & 0 \\ -\frac{2}{9} & -\frac{1}{3} & -\frac{A\alpha}{3} \\ \frac{4}{3A\alpha} & -\frac{2}{A\alpha} & -\frac{2}{3} \end{pmatrix} c_{23} \begin{pmatrix} -\frac{A\alpha}{2} \\ -\frac{2A\alpha}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.4.31)$$

Eqs. (5.4.28)-(5.4.30) then imply that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \rho \rightarrow \infty. \quad (5.4.32)$$

This second asymptotic behaviour corresponds to a general solution of a flat brane that is valid around a collapse-brane I singularity. We thus regain a behaviour similar to that of the balance in (5.3.25). However, an important outcome of our analysis is that there is now in addition to that previously met, an equally general distinct *big rip-brane singularity* that may also arise during the evolution of a flat brane.

5.4.2 The balance ${}_\gamma\mathcal{B}_2$

For the second balance ${}_\gamma\mathcal{B}_2$, we find that

$${}_\gamma\mathcal{K}_2 = D\mathbf{f}(\alpha, \alpha, 0) - \text{diag}(1, 0, -2) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -\frac{2}{3}A\alpha(1+2\gamma) \\ 0 & 0 & -2(1+2\gamma) \end{pmatrix}, \quad (5.4.33)$$

and hence its spectrum is

$$\text{spec}({}_\gamma\mathcal{K}_2) = \{-1, 0, -2(1+2\gamma)\}. \quad (5.4.34)$$

We note here that the third arbitrary constant will appear at $j = -2(1+2\gamma)$. After

substituting the forms

$$x = \sum_{j=0}^{\infty} c_{j1} \Upsilon^{j+1}, \quad y = \sum_{j=0}^{\infty} c_{j2} \Upsilon^j, \quad z = \sum_{j=0}^{\infty} c_{j3} \Upsilon^{j-2}, \quad (5.4.35)$$

in the system (5.4.5)-(5.4.7), we may try giving different values to γ : inserting $\gamma = -3/4$ in the system we meet the third arbitrary constant at $j = 1$ ($\text{spec}({}_{-3/4}\mathcal{K}_2) = \{-1, 0, 1\}$).

The variables (x, y, w) then expand in the following way:

$$x = \alpha \Upsilon + \frac{A\alpha}{6} c_{13} \Upsilon^2 + \dots, \quad (5.4.36)$$

$$y = \alpha + \frac{A\alpha}{3} c_{13} \Upsilon + \dots, \quad (5.4.37)$$

$$w = c_{13} \Upsilon^{-1} + \dots. \quad (5.4.38)$$

We ought to check the validity of the compatibility condition for $j = 1$. But this is again trivially satisfied since

$$({}_{-3/4}\mathcal{K}_2 - \mathcal{I}_3) \mathbf{c}_1 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & A\alpha/3 \\ 0 & 0 & 0 \end{pmatrix} c_{13} \begin{pmatrix} A\alpha/6 \\ A\alpha/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.4.39)$$

The series expansions Eqs. (5.4.36)-(5.4.38) are therefore valid ones, and we conclude that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \alpha, \quad \rho \rightarrow \infty, \quad \alpha \neq 0. \quad (5.4.40)$$

It will follow from the analysis below that the behaviour of ρ may vary according to our choice of γ .

Indeed for $\gamma = -1$ ($\text{spec}({}_{-1}\mathcal{K}_2) = \{-1, 0, 2\}$) we find that

$$x = \alpha \Upsilon + \frac{A\alpha}{9} c_{23} \Upsilon^3 + \dots, \quad (5.4.41)$$

$$y = \alpha + \frac{A\alpha}{3} c_{23} \Upsilon^2 + \dots, \quad (5.4.42)$$

$$w = c_{23} + \dots. \quad (5.4.43)$$

Note that the compatibility condition is satisfied also here:

$$({}_{-1}\mathcal{K}_2 - 2\mathcal{I}_3)\mathbf{c}_2 = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -2 & 2A\alpha/3 \\ 0 & 0 & 0 \end{pmatrix} c_{23} \begin{pmatrix} A\alpha/9 \\ A\alpha/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.4.44)$$

We see then that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \alpha, \quad \rho \rightarrow c_{23}, \quad \alpha \neq 0, \quad (5.4.45)$$

which shows that distinctly different behaviour than (5.4.40).

A yet different behaviour is met if we choose $\gamma = -5/4$: The \mathcal{K} -exponents are in this case given by $\text{spec}({}_{-5/4}\mathcal{K}_2) = \{-1, 0, 3\}$, and the series expansions read:

$$x = \alpha\Upsilon + \frac{A\alpha}{12}c_{33}\Upsilon^4 + \dots, \quad (5.4.46)$$

$$y = \alpha + \frac{A\alpha}{3}c_{33}\Upsilon^3 + \dots, \quad (5.4.47)$$

$$w = c_{33}\Upsilon + \dots. \quad (5.4.48)$$

These expansions are valid locally around the singularity since the compatibility condition holds true,

$$({}_{-5/4}\mathcal{K}_2 - 3\mathcal{I}_3)\mathbf{c}_1 = \begin{pmatrix} -4 & 1 & 0 \\ 0 & -3 & A\alpha \\ 0 & 0 & 0 \end{pmatrix} c_{32} \begin{pmatrix} A\alpha/12 \\ A\alpha/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.4.49)$$

For $\Upsilon \rightarrow 0$ we have that,

$$a \rightarrow 0, \quad a' \rightarrow \alpha, \quad \rho \rightarrow 0, \quad \alpha \neq 0. \quad (5.4.50)$$

This balance describes therefore the asymptotic behaviour of a particular solution of a curved brane around a finite-distance singularity where $a \rightarrow 0$ and $a' \rightarrow \alpha$, i.e., a and a'

behave exactly as in the case of a collapse-brane II singularity but now the behaviour of the energy density of the perfect fluid varies dramatically from zero to infinity ($\rho \rightarrow 0, \rho_s, \infty$) depending on the value of γ . This allows for new dynamical outcomes of the system (5.4.5)-(5.4.8): we can have a constant or even no flow of energy through the finite-distance singularity and into the extra dimension.

A totally different result is obtained when we consider a family of fluids having a γ greater than $-1/2$. For $\gamma = 0$, we have that $\text{spec}({}_0\mathcal{K}_2) = \{-1, 0, -2\}$ and hence we may expand (x, y, w) in descending powers in order to meet the arbitrary constant appearing at $j = -1$ and $j = -2$, i.e.,

$$x = \sum_{j=0}^{-\infty} c_{j1} \Upsilon^{j+1}, \quad y = \sum_{j=0}^{-\infty} c_{j2} \Upsilon^j, \quad z = \sum_{j=0}^{-\infty} c_{j3} \Upsilon^{j-2}. \quad (5.4.51)$$

We find:

$$x = \alpha \Upsilon + c_{-11} - \frac{A\alpha}{3} c_{-23} \Upsilon^{-1} + \dots, \quad (5.4.52)$$

$$y = \alpha + \frac{A\alpha}{3} c_{-23} \Upsilon^{-2} + \dots, \quad (5.4.53)$$

$$w = c_{-23} \Upsilon^{-4} + \dots. \quad (5.4.54)$$

The compatibility condition is satisfied since

$$({}_0\mathcal{K}_2 + 2\mathcal{I}_3)\mathbf{c}_{-2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -2A\alpha/3 \\ 0 & 0 & 0 \end{pmatrix} c_{-23} \begin{pmatrix} -A\alpha/3 \\ A\alpha/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.4.55)$$

As $S = 1/\Upsilon \rightarrow \infty$ we conclude that

$$a \rightarrow \infty, \quad a' \rightarrow \infty, \quad \rho \rightarrow \infty, \quad (5.4.56)$$

and we can therefore avoid the finite-distance singularity in this case.

5.4.3 The balances ${}_{-1/2}\mathcal{B}_3$, ${}_{-1/2}\mathcal{B}_4$ and ${}_{-1/2}\mathcal{B}_5$

In this Section, we analyse the behaviours that are described by the three balances which are valid once we set $\gamma = -1/2$ in the system (5.4.5)-(5.4.7). We show that the balance ${}_{-1/2}\mathcal{B}_3$, for $r < -2$, and the balance ${}_{-1/2}\mathcal{B}_4$ imply the existence of a collapse-brane II singularity. However, setting $r > -2$ in the balance ${}_{-1/2}\mathcal{B}_3$ leads to the avoidance of the finite-distance singularity. The balance ${}_{-1/2}\mathcal{B}_5$ on the other hand, describes a smooth behaviour since the warp factor, its derivative and the energy density of the fluid, remain finite asymptotically.

We begin with the analysis of the balance ${}_{-1/2}\mathcal{B}_3$. The \mathcal{K} -matrix for this balance is

$${}_{-1/2}\mathcal{K}_3 = D\mathbf{f}(\alpha, \alpha, 0) - \text{diag}(1, 0, r) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 - r \end{pmatrix}, \quad (5.4.57)$$

and hence,

$$\text{spec}({}_{-1/2}\mathcal{K}_3) = \{-1, 0, -2 - r\}. \quad (5.4.58)$$

Taking $-2 - r > 0$, we have two non-negative \mathcal{K} -exponents. For $r = -3$ we substitute the forms

$$x = \sum_{j=0}^{\infty} c_{j1} \Upsilon^{j+1}, \quad y = \sum_{j=0}^{\infty} c_{j2} \Upsilon^j, \quad z = \sum_{j=0}^{\infty} c_{j3} \Upsilon^{j-3}, \quad (5.4.59)$$

and arrive at the expansions

$$x = \alpha \Upsilon + \dots, \quad (5.4.60)$$

$$y = \alpha + \dots, \quad (5.4.61)$$

$$w = c_{13} \Upsilon^{-2} + \dots. \quad (5.4.62)$$

The compatibility condition is satisfied since

$$({}_{-1/2}\mathcal{K}_3 - \mathcal{I}_3)\mathbf{c}_1 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} c_{13} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.4.63)$$

The expansions (5.4.67)-(5.4.69) are therefore valid ones in the vicinity of the singularity and the general behaviour of the solution is characterised by

$$a \rightarrow 0, \quad a' \rightarrow \alpha, \quad \rho \rightarrow \infty, \quad \alpha \neq 0. \quad (5.4.64)$$

The balance ${}_{-1/2}\mathcal{B}_3$ for $r < -2$ implies therefore the existence of a collapse-brane II singularity.

If on the other hand we take $r > -2$, we have two negative \mathcal{K} -exponents. If we choose the value $r = 0$, then the spectrum is found to be

$$\text{spec}({}_{-1/2}\mathcal{K}_3) = \{-1, 0, -2\}, \quad (5.4.65)$$

and so inserting the forms

$$x = \sum_{j=0}^{-\infty} c_{j1} \Upsilon^{j+1}, \quad y = \sum_{j=0}^{-\infty} c_{j2} \Upsilon^j, \quad z = \sum_{j=0}^{-\infty} c_{j3} \Upsilon^j, \quad (5.4.66)$$

we obtain

$$x = \alpha \Upsilon + c_{-11}, \quad (5.4.67)$$

$$y = \alpha, \quad (5.4.68)$$

$$w = c_{-23} \Upsilon^{-2} + \dots, \quad (5.4.69)$$

which validates the compatibility condition at $j = -2$ since

$$({}_{-1/2}\mathcal{K}_3 + 2\mathcal{I}_3)\mathbf{c}_{-2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} c_{-23} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.4.70)$$

We see that as $S = 1/\Upsilon \rightarrow \infty$,

$$a \rightarrow c_{-11}, \quad a' \rightarrow \alpha, \quad \rho \rightarrow \infty, \quad \alpha \neq 0, \quad (5.4.71)$$

so that the balance ${}_{-1/2}\mathcal{B}_3$ for $r > -2$ offers the possibility of avoiding the finite-distance singularity met previously.

The balance ${}_{-1/2}\mathcal{B}_4$ is one with

$${}_{-1/2}\mathcal{K}_4 = D\mathbf{f}(\alpha, \alpha, 0) - \text{diag}(1, 0, -2) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ \frac{2\delta}{\alpha} & -\frac{2\delta}{\alpha} & 0 \end{pmatrix}, \quad (5.4.72)$$

and

$$\text{spec}({}_{-1/2}\mathcal{K}_4) = \{-1, 0, 0\}. \quad (5.4.73)$$

We note that the double multiplicity of the zero eigenvalue reflects the fact that there were already two arbitrary constants, α and δ in this balance (δ will have to be later sacrificed in order for that balance to satisfy the constraint Eq. (5.4.8)). We can thus write

$$x = \alpha\Upsilon + \dots, \quad (5.4.74)$$

$$y = \alpha + \dots, \quad (5.4.75)$$

$$w = \delta\Upsilon^{-2} + \dots, \quad (5.4.76)$$

so that as $\Upsilon \rightarrow 0$ a collapse-brane II singularity arises, i.e.,

$$a \rightarrow 0, \quad a' \rightarrow \alpha, \quad \rho \rightarrow \infty, \quad \alpha \neq 0. \quad (5.4.77)$$

Finally we come to the balance ${}_{-1/2}\mathcal{B}_5 = \{(\alpha, 0, 0), (0, -1, r)\}$. This has

$${}_{-1/2}\mathcal{K}_5 = D\mathbf{f}(\alpha, 0, 0) - \text{diag}(0, -1, r) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r \end{pmatrix}, \quad (5.4.78)$$

and

$$\text{spec}({}_{-1/2}\mathcal{K}_5) = \{1, 0, -r\}, \quad (5.4.79)$$

so we shall have to set $r = 1$ in order to have the -1 eigenvalue corresponding to the arbitrary position of the singularity, t_s . After substitution of the forms

$$x = \sum_{j=0}^{-\infty} c_{j1} \Upsilon^j, \quad y = \sum_{j=0}^{-\infty} c_{j2} \Upsilon^{j-1}, \quad z = \sum_{j=0}^{-\infty} c_{j3} \Upsilon^{j+1}, \quad (5.4.80)$$

we find that

$$x = \alpha + c_{11} + \cdots, \quad (5.4.81)$$

$$y = c_{11} + \cdots, \quad (5.4.82)$$

$$w = 0 + \cdots. \quad (5.4.83)$$

The compatibility condition is satisfied since

$$({}_{-1/2}\mathcal{K}_5 - \mathcal{I}_3)\mathbf{c}_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} c_{11} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (5.4.84)$$

and we see that as $\Upsilon \rightarrow 0$,

$$x \rightarrow \alpha, \quad y \rightarrow c_{11}, \quad w \rightarrow 0, \quad \alpha \neq 0. \quad (5.4.85)$$

5.5 Conclusions

The following two tables summarise the overall features of a braneworld consisting of a flat, or curved three-brane embedded in a five-dimensional bulk space with a scalar field (Table 5.1), or a perfect fluid (Table 5.2). In what follows, we denote by f_s the non-zero constant that is the limit of a function f on approach to the singularity at Y_s , and by FDS, IDS a finite-distance and an infinite-distance singularity respectively.

Table 5.1: Singularities in the case of a scalar field in the bulk

Balance	k	Type of singularity
\mathcal{B}_1	$k = 0$	FDS: $a \rightarrow 0, a' \rightarrow \infty, \phi \rightarrow \infty$
\mathcal{B}_2	$k \neq 0$	IDS: $a \rightarrow \infty, a' \rightarrow \infty, \phi \rightarrow \infty$

Table 5.2: Singularities in the case of a perfect fluid in the bulk

Balance	k	γ	r	Type of singularity
${}_\gamma\mathcal{B}_1$	$k = 0$	$\gamma > -1/2$	-	FDS: $a \rightarrow 0, a' \rightarrow \infty, \rho \rightarrow \infty$
		$\gamma < -1$	-	FDS: $a \rightarrow \infty, a' \rightarrow \infty, \rho \rightarrow \infty$
${}_\gamma\mathcal{B}_2$	$k \neq 0$	$\gamma < -1/2$	-	FDS: $a \rightarrow 0, a' \rightarrow \alpha, \rho \rightarrow 0, \rho_s, \infty$
		$\gamma > -1/2$	-	IDS: $a \rightarrow \infty, a' \rightarrow \infty, \rho \rightarrow \infty$
${}_{-1/2}\mathcal{B}_3$	$k \neq 0$	$\gamma = -1/2$	$r < -2$	FDS: $a \rightarrow 0, a' \rightarrow \alpha, \rho \rightarrow \infty$
			$r > -2$	IDS: $a \rightarrow \infty, a' \rightarrow \infty, \rho \rightarrow \infty$
${}_{-1/2}\mathcal{B}_4$	$k = 0, \pm 1$	$\gamma = -1/2$	-	FDS: $a \rightarrow 0, a' \rightarrow \alpha, \rho \rightarrow \infty$

For a flat brane and for both forms of bulk matter we find that there exists a collapse-brane I singularity as this is implied by the balances \mathcal{B}_1 and ${}_\gamma\mathcal{B}_1$ with $\gamma > -1/2$. When we consider a scalar field this is the *only* possible type of FDS that may arise. In contrast, when a perfect fluid is present in the bulk there are in addition two new types of FDS: The first one is a very distinct big rip-brane singularity that appears only when a phantom equation of state is considered ($\gamma < -1$). This new possibility is offered by the balance ${}_\gamma\mathcal{B}_1$ and corresponds to a general solution. The second new type of FDS is that of a collapse-brane II singularity that is implied by the balance ${}_{-1/2}\mathcal{B}_4$. However this feature is less general than the ones described by the balance ${}_\gamma\mathcal{B}_1$. This follows not only because we had to fix the value of γ equal to $-1/2$ in the balance ${}_{-1/2}\mathcal{B}_4$, but also because we had

to sacrifice one arbitrary constant by setting $\delta = 3/(2A)$ in order to satisfy the constraint Eq. (5.4.8). This implies that ${}_{-1/2}\mathcal{B}_4$ describes a particular solution with two arbitrary constants.

For a non-flat brane on the other hand, the dynamical evolution of the braneworlds under consideration depends strongly on the form of their bulk matter: although FDSs *cannot* arise when a scalar field fills the bulk, they *do* become possible once this field is replaced with a perfect fluid. In particular, both ${}_{-1/2}\mathcal{B}_4$ and ${}_{-1/2}\mathcal{B}_3$, with $r < -2$ offer the possibility of a collapse-brane II singularity, while the balance ${}_{\gamma}\mathcal{B}_2$ with $\gamma < -1/2$ implies the existence of a FDS around which $a \rightarrow 0$ and $a' \rightarrow \alpha$, i.e., a and a' behave exactly as in the case of a collapse-brane II singularity but the energy leak into the extra dimension is monitored each time by the γ parameter that defines the type of fluid (or the dominant exponent). However, for a non-flat brane there still exists the possibility of escaping the FDSs and this is suggested by the balances ${}_{\gamma}\mathcal{B}_2$ with $\gamma > -1/2$ and ${}_{-1/2}\mathcal{B}_3$ with $r > -2$. (Recall that all possible balances in the case of a non-flat brane correspond to particular solutions with two arbitrary constants.)

It would be interesting to further investigate if the FDSs found here continue to emerge when we consider the case in which the scalar field *coexists* with the perfect fluid in the bulk. The analysis of this more involved case may also shed light to the factors that control how these two bulk matter components compete on approach to the singularity or even predict new types of FDSs that might then become feasible.

Chapter 6

Conclusions and future work

Two main themes throughout this thesis were completeness and singularities. In particular, in Chapter 2 we stated and proved the general completeness theorem of [72] for the case of an isotropic spacetime. This theorem indicates the underlying reasons that lead to completeness in the category of isotropic models, some examples of which we analysed later on in that Chapter.

We continued in Chapter 3 by formulating necessary conditions for the appearance of finite-time singularities in isotropic universes based entirely on the behaviour of the Hubble parameter. These conditions provided us with a first classification of such singularities, a scheme which was then further refined and expanded with the use of the Bel-Robinson energy. This energy represents the energy of the gravitational field projected in a sense to a slice in spacetime and it has been previously used for the proof of global-in-time results (see [206], [207], [208]). Combining the quantities that constitute the Bel Robinson energy with the pressure and density of the matter fields through the field equations, we were thus led to a clear picture of how the matter fields influence the nature of the singularities. The new classification scheme is based on the behaviour of the Hubble parameter, the Bel-Robinson energy and the scale factor. As it was shown later on in Chapter 4, an examination of models from the recent literature illustrates that this classification not only includes all known types of singularities but it also predicts new ones. We believe that

this classification is complete in the isotropic category in the sense that it depicts all the different qualities in which singularities may differ. Chapter 3 ended with the formulation of a relation that holds between the existence of a closed trapped surface and the divergence of the Bel-Robinson energy; this relation provided us with the information needed to decide on the fate of isotropic cosmologies and we illustrated this important result with various examples.

In Chapter 4, we analysed under the prism of our classification a wide variety of isotropic cosmologies that accommodate during their existence one or even more of the four highlighted singularity types of our classification scheme, namely a collapse, big-rip, sudden, or a sudden-rip singularity. There are many more types predicted by our classification and the search for adequate physical circumstances giving rise to these types of singularities is a project of future work.

Incorporating these new singularities in the evolution of particular cosmologies will lead to a precise understanding of the role that these singularities may play in determining the overall destiny of such cosmologies. Recall from Section 4.3 that it was only after observations indicated that dark energy may be described by the phantom equation of state that the possibility of a big-rip singularity was considered seriously for the first time. This led to the discovery that our universe may face a violent end on approach to which every bound structure such as galaxies, stars, planetary systems or even atoms would be ripped apart.

A similar situation may arise in the consideration of models accommodating the not-so-far studied singularity types of our classification: in the case that some model comes in agreement with observations we may be able to make predictions about the future evolution of the observed universe.

Another project of future work is to extend the present classification by first considering anisotropic and then inhomogeneous spacetimes within the framework of general relativity; we have already discussed in Chapter 3 (see Section 3.5) the way that the classification scheme presented in this work is expected to be enriched with singularities that are by

necessity impossible to occur in isotropic models.

A step towards another direction is to examine how our classification does change when we abandon general relativity and consider other metric theories of gravity such as $f(R)$ theories or scalar-tensor gravity. With this examination we will be able to recognise the way that these different theories of gravity affect the number and types of singularities that are possible to occur in a given class of spacetimes.

The main body of this thesis ended with Chapter 5 at which point we took a turn from the classical case of a four-dimensional spacetime to explore the character of singularities that arise in the different cosmological landscape of braneworlds. Our research focused on the asymptotic analysis of singularities that may arise during the dynamical evolution of a three-brane embedded in a five-dimensional bulk space that is inhabited by a scalar field or a perfect fluid. Our analysis, which was performed with the method of asymptotic splittings of [76], shows that there are three main possible types of finite-distance singularities which we call collapse-brane I, II and big rip-brane singularities. A common characteristic of these singularities is that they act as a reservoir through which all the energy density of the bulk-matter components may be emptied.

This phenomenon of an infinite leak of energy into the extra dimension was previously used in [74] as an explanation to the cosmological constant problem. Their model consisted of a flat three-brane embedded in a five dimensional bulk with a scalar field and exhibited a collapse-brane I singularity that successfully cut off the extra dimension and swallowed the vacuum energy, offering thus an explanation for the observed four-dimensional flatness and the vanishing value of the cosmological constant. We showed that this singularity is removed to an infinite distance from the brane when instead of a flat brane we considered a curved one while by altering the matter in the bulk we allowed for the new distinct types of finite-distance singularities mentioned above to emerge.

Future work in the field of braneworld singularities includes the construction of new versions of our model with the consideration of other forms of bulk matter such as the combination of a scalar field and a perfect fluid, or a fluid with unconnected pressure and

density. Studying different forms of bulk matter may lead to the discovery of new finite-distance singularities which will help us to draw conclusions about the possible variety of singularities that these braneworlds can accommodate and hence help to determine their true range of validity.

Appendix A

Notation and conventions

In this thesis we use the signature $(-, +, +, +)$ for the metric tensor g_{ab} . The Riemann tensor is defined by

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{cf} \Gamma^f{}_{bd} - \Gamma^a{}_{fd} \Gamma^f{}_{bc}, \quad (\text{A.0.1})$$

where ∂_a stands for the partial derivative $\partial/\partial x^a$ and

$$\Gamma^a{}_{bd} = \frac{1}{2} g^{ac} (\partial_b g_{cd} + \partial_d g_{cb} - \partial_c g_{bd}). \quad (\text{A.0.2})$$

are the Christoffel symbols. The Ricci tensor is defined by contracting the first and the third index of the Riemann tensor, i.e.,

$$R_{ab} = R^c{}_{acb}. \quad (\text{A.0.3})$$

The Einstein tensor is given by

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \quad (\text{A.0.4})$$

where R is the scalar curvature, $R = g^{ab} R_{ab}$.

In Chapters 2,3,4 we use units so that $8\pi G = c = 1$. The Einstein equations then read

$$G_{ab} = +T_{ab} \tag{A.0.5}$$

where T_{ab} is the energy-momentum tensor.

The symbol ∇_a stands for the covariant derivative.

We also use the following abbreviations: RW for Robertson-Walker, FRW for Friedmann-Robertson-Walker, CTS for closed trapped surface, FDS for finite-distance singularity and IDS for infinite-distance singularity.

Appendix B

Some submanifold geometry formulae

In this appendix we review some useful formulae from submanifold geometry. In Section B.1, we calculate the extrinsic curvature of the spacelike slices of an arbitrary globally hyperbolic spacetime. In Subsection B.1.1, we show that the extrinsic curvature of an RW space is proportional to the Hubble rate. In Section B.2, we derive the basic Gauss-Codazzi relations that associate the Riemann and the Ricci tensor of the spacelike slices with the Riemann and the Ricci tensor of the spacetime.

B.1 Extrinsic curvature

Consider a globally hyperbolic spacetime (\mathcal{V}, g) , that is a spacetime which can be foliated by Cauchy surfaces $\Sigma_t \equiv \Sigma \times \{t\}$ parametrised by a global time function t , and a smooth congruence of timelike geodesics in (\mathcal{V}, g) orthogonal to the spacelike hypersurfaces Σ_t . If n^a is the unit tangent vector field ($n^a n_a = -1$) to these geodesics, then n^a is also the unit normal vector field to the hypersurfaces Σ_t . The spacetime metric g_{ab} induces the spatial metric h_{ab} on Σ_t given by [77]

$$h_{ab} = g_{ab} + n_a n_b. \tag{B.1.1}$$

It then follows that h_{ab} satisfies

$$h_{ab}n^a = 0, \quad h_{ab}h_c^b = h_{ac}. \quad (\text{B.1.2})$$

To understand the role that h_{ab} plays consider a spacetime vector w^a . We can split w^a into components tangent and normal to Σ_t so that

$$w^a = w_\perp n^a + w_\parallel^a, \quad (\text{B.1.3})$$

where $w_\parallel^a n_a = 0$. If w^a is a tangential vector, then $w^a = w_\parallel^a$ and $w_\perp = 0$, or equivalently

$$w^a = h_b^a w^b. \quad (\text{B.1.4})$$

This condition can be generalised to apply to an arbitrary tensor in the following way: we say that the tensor $T_{b_1 \dots b_l}^{a_1 \dots a_k}$ is tangential to Σ_t if

$$T_{b_1 \dots b_l}^{a_1 \dots a_k} = h_{c_1}^{a_1} \dots h_{c_k}^{a_k} h_{b_1}^{d_1} \dots h_{b_l}^{d_l} T_{d_1 \dots d_l}^{c_1 \dots c_k}. \quad (\text{B.1.5})$$

This means that h_b^a is the projection operator onto the tangential plane perpendicular to n^a .

The extrinsic curvature K_{ab} of Σ_t is defined as

$$K_{ab} = \nabla_a n_b. \quad (\text{B.1.6})$$

Therefore the extrinsic curvature describes the *bending* of Σ_t in the spacetime. To see this more clearly, consider the orthogonal deviation vector ξ^a representing the spatial infinitesimal displacement between two neighboring geodesics of the congruence. Then $\mathcal{L}_n \xi^a = 0$, which means that $n^b \nabla_b \xi^a = \xi^b \nabla_b n^a = K_b^a \xi^b$. The last equality implies that the extrinsic curvature measures the failure of n^a to be parallelly transported along a spacelike geodesic with tangent vector ξ^a . In the same time, the first equality says that this failure is equal to the failure of ξ^a to be parallelly transported along the timelike geodesic with tangent

n^a . This means that an observer on one of the geodesics would see a nearby geodesic being stretched and twisted [209].

Note that K_{ab} is purely spatial because $K_{ab}n^b = 0$. Since we have considered a hypersurface-orthogonal congruence of geodesics, it can be proved (cf. [77], p. 217) that the rotation ω_{ab} , defined by $\omega_{ab} = \frac{1}{2}(\nabla_b n_a - \nabla_a n_b) = K_{[ba]}$, is zero and hence the extrinsic curvature tensor is symmetric $K_{ab} = K_{ba}$. We can therefore write

$$2K_{ab} = \nabla_a n_b + \nabla_b n_a. \quad (\text{B.1.7})$$

The right hand side of the above equation equals the Lie derivative of g_{ab} with respect to n^a , hence

$$K_{ab} = \frac{1}{2}\mathcal{L}_n g_{ab} = \frac{1}{2}\mathcal{L}_n(h_{ab} - n_a n_b) = \frac{1}{2}\mathcal{L}_n h_{ab} - \frac{1}{2}\mathcal{L}_n(n_a n_b). \quad (\text{B.1.8})$$

But

$$\mathcal{L}_n(n_a)n_b = (n^c \nabla_c n_a + n_c \nabla_a n^c)n_b = \left(g_{ad} n^c \nabla_c n^d + \frac{1}{2} \nabla_a(n_c n^c) \right) n_b = 0, \quad (\text{B.1.9})$$

because of the geodesic equation $n^c \nabla_c n^d = 0$, and the fact that $n^c n_c = -1$. Using this and the Leibnitz rule in Eq. (B.1.8), we find the following formula for the extrinsic curvature:

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}. \quad (\text{B.1.10})$$

Since a Gaussian coordinate system is adapted to n^a , we finally get

$$K_{ab} = \frac{1}{2} \frac{\partial h_{ab}}{\partial t}. \quad (\text{B.1.11})$$

This relation states that the extrinsic curvature is the rate of change of the spatial metric h_{ab} along the congruence.

B.1.1 Extrinsic curvature for a RW space

Consider now a RW metric, i.e., one with

$$ds^2 = -dt^2 + a^2(t) (dr^2 + f^2(r)(d\theta^2 + \sin^2 \theta d\phi^2)), \quad (\text{B.1.12})$$

where

$$f(r) = \begin{cases} r, & k = 0 \\ \sin r, & k = 1 \\ \sinh r, & k = -1. \end{cases} \quad (\text{B.1.13})$$

Then by the use of Eq. (B.1.11), we find

$$\begin{aligned} K_{11} &= a(t)\dot{a}(t), \\ K_{22} &= f^2(r)a(t)\dot{a}(t), \\ K_{33} &= f^2(r)\sin^2 \theta a(t)\dot{a}(t). \end{aligned}$$

Now, since the spatial norm of a 2-covariant tensor X_{ij} is given by

$$|X|^2 = g^{ij}g^{kl}X_{ik}X_{jl} \quad i, j, k, l = 1, 2, 3, \quad (\text{B.1.14})$$

we find that the norm of the extrinsic curvature is given by

$$|K|^2 = g^{ij}g^{rs}K_{ir}K_{js} = (g^{11})^2K_{11}^2 + (g^{22})^2K_{22}^2 + (g^{33})^2K_{33}^2 = 3\frac{\dot{a}^2(t)}{a^2(t)} = 3H^2(t), \quad (\text{B.1.15})$$

where $H(t)$ is the Hubble expansion rate, $H = \dot{a}/a$.

B.2 Gauss-Codazzi formalism

As we saw in the previous Section, the metric g_{ab} induces a Riemannian metric h_{ab} on each hypersurface Σ_t . We know that there exists a unique derivative operator associated with

this metric, which we denote by D_c . It can be proved that D_c is given by [77]

$$D_c T_{b_1 \dots b_l}^{a_1 \dots a_k} = h_{d_1}^{a_1} \dots h_{b_l}^{e_l} h_c^f \nabla_f T_{e_1 \dots e_l}^{d_1 \dots d_k}. \quad (\text{B.2.1})$$

In turn D_c determines a curvature tensor ${}^{(3)}R_{abc}^d$ on Σ_t . This tensor is defined in analogy with the four dimensional case by the relation

$$D_a D_b u_c - D_b D_a u_c = {}^{(3)}R_{abc}^d u_d. \quad (\text{B.2.2})$$

Our goal is to find the relation between the Riemann tensor of the hypersurface ${}^{(3)}R_{abc}^d$ and the Riemann tensor of the whole spacetime R_{abc}^d (Eq. B.2.12 below). We start by calculating $D_a D_b u_c$:

$$\begin{aligned} D_a D_b u_c &= D_a (h_c^e h_b^d \nabla_d u_e) = h_b^g h_c^k h_a^f \nabla_f (h_k^e h_g^d \nabla_d u_e) \\ &= h_b^g h_c^k h_k^e h_g^d h_a^f (\nabla_f \nabla_d u_e) + h_g^d h_b^g h_c^k h_a^f (\nabla_d u_e) \nabla_f h_k^e \\ &+ h_b^g h_c^k h_a^f h_k^e (\nabla_d u_e) (\nabla_f h_g^d). \end{aligned} \quad (\text{B.2.3})$$

But by the use of

$$\begin{aligned} h_a^f h_c^k \nabla_f h_k^e &= h_a^f h_c^k \nabla_f (g_k^e + n_k n^e) \\ &= h_a^f h_c^k (\nabla_f n_k) n^e + h_a^f h_c^k n_k \nabla_f n^e + h_a^f h_c^k \nabla_f g_k^e \\ &= h_c^k K_{ak} n^e = K_{ac} n^e, \end{aligned} \quad (\text{B.2.4})$$

Eq. (B.2.3) becomes,

$$D_a D_b u_c = h_b^g h_g^d h_c^k h_k^e h_a^f (\nabla_f \nabla_d u_e) + K_{ac} n^e h_b^d (\nabla_d u_e) + h_c^e K_{ab} n^d (\nabla_d u_e). \quad (\text{B.2.5})$$

Now since

$$h_b^d (\nabla_d u_e) n^e = h_b^d \nabla_d (u_e n^e) - h_b^d (\nabla_d n^e) u_e = -K_b^e u_e, \quad (\text{B.2.6})$$

we find for $D_a D_b u_c$,

$$D_a D_b u_c = h_a^f h_b^d h_c^e \nabla_f \nabla_d u_e + h_c^e K_{ab} n^d \nabla_d u_e - K_b^e u_e K_{ac}. \quad (\text{B.2.7})$$

Similarly, we get an analogous expression for $D_b D_a u_c$,

$$D_b D_a u_c = h_b^d h_a^f h_c^e \nabla_d \nabla_f u_e + h_c^e K_{ba} n^d \nabla_d u_e - K_a^e u_e K_{bc}. \quad (\text{B.2.8})$$

Subtracting Eqs. (B.2.3) and (B.2.8) we have that

$$\begin{aligned} D_a D_b u_c - D_b D_a u_c &= -K_{ac} K_b^e u_e + K_{bc} K_a^e u_e + h_a^f h_b^d h_c^e (\nabla_f \nabla_d u_e - \nabla_d \nabla_f u_e) \\ &= -K_{ac} K_b^e h_e^d u_d + K_{bc} K_a^e h_e^d u_d + h_a^f h_b^d h_c^e R_{fde}^j u_j \\ &= -K_{ac} K_b^d u_d + K_{bc} K_a^d u_d + h_a^f h_b^d h_c^e h_j^d R_{fde}^j u_d. \end{aligned} \quad (\text{B.2.9})$$

Then from Eq. (B.2.9) and Eq. (B.2.2) follows the first Gauss-Codazzi relation:

$${}^{(3)}R_{abc}^d = h_a^f h_b^d h_c^e h_j^d R_{fde}^j - K_{ac} K_b^d + K_{bc} K_a^d. \quad (\text{B.2.10})$$

In Gaussian normal coordinates it holds that

$$h_a^f = h^{f\lambda} h_{\lambda a} = \delta_a^f, \quad (\text{B.2.11})$$

so that Eq. (B.2.10) becomes

$${}^{(3)}R_{abc}^d = R_{abc}^d - K_{ac} K_b^d + K_{ab} K_c^d, \quad (\text{B.2.12})$$

where we have used the fact that $K_{bc} K_a^d = g_{lc} g^{la} K_{ab} K_a^d = K_{ab} K_c^d$.

Now we can use the Eq. (B.2.12) in order to deduce the analogous relation that holds between the Ricci tensor of the hypersurface ${}^{(3)}R_{ab}$ and the Ricci tensor of the spacetime,

R_{ab} . For that purpose, we start by calculating the component R_{anb}^n (cf. [210], pp. 163-164):

$$\begin{aligned}
R_{arb}^d n_d n^r h_q^b h_m^a &= (\nabla_a \nabla_r n_b - \nabla_r \nabla_a n_b) n^r h_q^b h_m^a = (\nabla_a K_{rb}) n^r h_q^b h_m^a - (\nabla_r K_{ab}) n^r h_q^b h_m^a \\
&= \nabla_a (K_{rb} n^r h_q^b h_m^a) - K_{rb} (\nabla_a n^r) h_q^b h_m^a - K_{rb} n^r \nabla_a (h_q^b h_m^a) \\
&- (\mathcal{L}_n K_{ab} - K_{rb} \nabla_a n^r - K_{ar} \nabla_b n^r) h_q^b h_m^a \\
&= (K_{ar} K_b^r - \mathcal{L}_n K_{ab}) h_q^b h_m^a,
\end{aligned} \tag{B.2.13}$$

so that

$$R_{arb}^d n_d n^r = K_{ar} K_b^r - \mathcal{L}_n K_{ab}. \tag{B.2.14}$$

Introducing Gaussian-normal coordinates and contracting the first and the third index, we have that

$$R_{anb}^n = -K_{ar} K_b^r + \frac{\partial K_{ab}}{\partial n}. \tag{B.2.15}$$

The Ricci tensor is then given by

$$R_{ab} = R_{amb}^m + R_{anb}^n = {}^{(3)}R_{ab} + K_{ab} K - 2K_{an} K_b^n + \frac{\partial K_{ab}}{\partial n}. \tag{B.2.16}$$

We can work in a similar way for the case of a timelike hypersurface with normal vector n^a (now $n^a n_a = 1$). Both cases of a spacelike and timelike normal vector can be treated at the same time throughout the previous calculations by using the symbol $\epsilon(n) = n^a n_a = \pm 1$. In particular, with the use of ϵ , the key equations (B.1.1), (B.2.12), (B.2.14) and (B.2.16) can be written in the following form:

$$h_{ab} = g_{ab} - \epsilon(n) n_a n_b, \tag{B.2.17}$$

$${}^{(3)}R_{abc}^d = R_{abc}^d + \epsilon(n) (K_{ac} K_b^d - K_{ab} K_c^d), \tag{B.2.18}$$

$$R_{anb}^n \epsilon(n) = K_{ar} K_b^r - \mathcal{L}_n K_{ab}, \tag{B.2.19}$$

$$R_{ab} = {}^{(3)}R_{ab} + \epsilon(n) \left(-K_{ab} K + 2K_{an} K_b^n - \frac{\partial K_{ab}}{\partial n} \right). \tag{B.2.20}$$

Appendix C

Bel-Robinson energy

This appendix deals with the notion of the Bel-Robinson energy. In Section C.1, we give the definition of the Bel-Robinson energy for the general form of a sliced space and in Subsection C.1.1, we calculate the Bel-Robinson energy of a RW space. In the end of the Subsection C.1.1, we show that the null energy condition in closed or flat accelerating universes translates into an inequality satisfied by the electric parts of the Bel-Robinson energy.

C.1 Definition

The Bel-Robinson energy and related quantities play an important role in proving global properties of spacetimes. For example, they were used in [208], [207] to prove global existence results in the case of asymptotically flat and cosmological spacetimes respectively. In this thesis, the Bel-Robinson energy is used for the purposes of classification of spacetime singularities.

The Bel-Robinson energy is a kind of energy of the gravitational field *projected* in a sense to a slice in spacetime. Consider the sliced spacetime with metric

$${}^{(3+1)}g \equiv -N^2(\theta^0)^2 + g_{ij} \theta^i \theta^j, \quad \theta^0 = dt, \quad \theta^i \equiv dx^i + \beta^i dt, \quad (\text{C.1.1})$$

where $N = N(t, x^i)$ is the lapse function and $\beta^i(t, x^j)$ is the shift function. The 2-covariant spatial electric and magnetic tensors are defined as follows

$$E_{ij} = R_{i0j}^0, \quad (\text{C.1.2})$$

$$D_{ij} = \frac{1}{4} \eta_{ihk} \eta_{jlm} R^{hklm}, \quad (\text{C.1.3})$$

$$H_{ij} = \frac{1}{2} N^{-1} \eta_{ihk} R_{0j}^{hk}, \quad (\text{C.1.4})$$

$$B_{ji} = \frac{1}{2} N^{-1} \eta_{ihk} R_{0j}^{hk}, \quad (\text{C.1.5})$$

where η_{ijk} is the volume element of the space metric $g_t \equiv g_{ij} dx^i dx^j$. These four time-dependent space tensors comprise what is called a *Bianchi field*, (E, H, D, B) , a very important frame field used to prove global in time results, cf. [94]. The *Bel-Robinson energy at time t* is given by

$$\mathcal{B}(t) = \frac{1}{2} \int_{\mathcal{M}_t} (|E|^2 + |D|^2 + |B|^2 + |H|^2) d\mu_{g_t}, \quad (\text{C.1.6})$$

where by $|X|^2 = g^{ij} g^{kl} X_{ik} X_{jl}$ we denote the spatial norm of the 2-covariant tensor X .

C.1.1 Bel-Robinson energy of an isotropic universe

Consider the RW metric (2.1.2). In order to compute the Bel-Robinson energy at time t , $\mathcal{B}(t)$, we first calculate the two covariant spatial tensors E_{ij} and D_{ij} using relations (C.1.2) and (C.1.3). These relations give

$$\begin{aligned} E_{11} &= a\ddot{a}, \\ E_{22} &= f^2 a\ddot{a}, \\ E_{33} &= f^2 \sin^2 \theta a\ddot{a}, \end{aligned}$$

while

$$\begin{aligned}
D_{11} &= f^{-2}(1 - f'^2 + f^2\dot{a}^2), \\
D_{22} &= f^2 \left(\dot{a}^2 - \frac{f''}{f} \right), \\
D_{33} &= f^2 \sin^2 \theta \left(\dot{a}^2 - \frac{f''}{f} \right),
\end{aligned}$$

where a prime denotes the differentiation of $f \equiv f(r)$ with respect to r , while a dot denotes the differentiation of $a \equiv a(t)$ with respect to t . The magnetic tensors H_{ij} , B_{ji} make no contribution to the Bel-Robinson energy since for a RW metric we have that $R_{0j}^{hk} = 0$ for $h, k, j = 1, 2, 3$.

Next, we make use of Eq. (B.1.14) in order to calculate the spatial norms of the two electric tensors. We find

$$|E|^2 = (g^{11})^2 E_{11}^2 + (g^{22})^2 E_{22}^2 + (g^{33})^2 E_{33}^2 = 3 \frac{\ddot{a}^2}{a^2}, \quad (\text{C.1.7})$$

and

$$|D|^2 = (g^{11})^2 D_{11}^2 + (g^{22})^2 D_{22}^2 + (g^{33})^2 D_{33}^2 = \left(\frac{1 - f'^2}{a^2 f^2} + \frac{\dot{a}^2}{a^2} \right)^2 + 2 \left(\frac{\dot{a}^2}{a^2} - \frac{f''}{f a^2} \right)^2. \quad (\text{C.1.8})$$

It then follows from the form of $f(r)$ (cf. Eq. (2.1.3)) that

$$\frac{1 - f'^2}{f^2} = -\frac{f''}{f} = k, \quad (\text{C.1.9})$$

where k is the constant curvature of the spatial slice. We can therefore write $|D|$ in the form

$$|D|^2 = 3 \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right)^2. \quad (\text{C.1.10})$$

The Bel-Robinson energy at time t is then given by

$$\mathcal{B}(t) = \frac{C}{2} (|E|^2 + |D|^2) = \frac{3C}{2} \left(\frac{\ddot{a}^2}{a^2} + \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right)^2 \right), \quad (\text{C.1.11})$$

where C is the constant volume of (or in, in the case of a non-compact space) the three dimensional slice at time t .

Finally, we have the following proposition that translates the null energy condition into an inequality satisfied by the electric parts of the Bel-Robinson energy.

Proposition C.1.1 *In an accelerating flat or closed universe the null energy condition is equivalent to the inequality $|E| \leq |D|$.*

PROOF. The null energy condition demands that

$$R_{ab}k^ak^b \geq 0, \quad \text{for every null vector } k^a. \quad (\text{C.1.12})$$

In an isotropic universe with metric described by Eqs. (2.1.2) and (2.1.3), the above condition translates to

$$\begin{aligned} R_{ab}k^ak^b &= R_{11}(k^1)^2 + R_{22}(k^2)^2 + R_{33}(k^3)^2 + R_{44}(k^4)^2 \\ &= \left(2\dot{a}^2 - 2\frac{f''}{f} + a\ddot{a} \right) (k^1)^2 + (1 - f'^2 - ff'' + f^2(2\dot{a}^2 + a\ddot{a})) (k^2)^2 + \\ &\quad + \sin^2\theta(1 - f'^2 - ff'' + f^2(2\dot{a}^2 + a\ddot{a})) (k^3)^2 - 3\frac{\ddot{a}}{a}(k^4)^2. \end{aligned} \quad (\text{C.1.13})$$

Keeping in mind that $1 - f'^2 = -ff''$, we can write Eq. (C.1.13) as follows

$$\begin{aligned} R_{ab}k^ak^b &= \left(2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} - 2\frac{f''}{fa^2} \right) [a^2((k^1)^2 + f^2((k^2)^2 + \sin^2\theta(k^3)^2)) - (k^4)^2] + \\ &\quad + \left(-3\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} - 2\frac{f''}{fa^2} \right) (k^4)^2 = \left(-2\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - 2\frac{f''}{fa^2} \right) (k^4)^2. \end{aligned}$$

Therefore if the null energy condition, Eq. (C.1.12), holds then

$$-\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \geq 0, \quad (\text{C.1.14})$$

where we have used the fact that $-f''/f = k$, with k being the constant curvature of the spatial slices. Since we have assumed a $k \geq 0$ accelerating universe, the electric parts are given by

$$\begin{aligned} |D| &= \sqrt{3} \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right), \\ |E| &= \sqrt{3} \frac{\ddot{a}}{a} \end{aligned}$$

hence Eq.(C.1.14) implies that $|E| \leq |D|$. ■

Appendix D

Energy-momentum tensor of a scalar field

In order to define the energy-momentum tensor for a scalar field we start from the action

$$S_\phi = \int \mathcal{L}_\phi \sqrt{-g} d^4x, \quad (\text{D.0.1})$$

where the Lagrangian density is

$$\mathcal{L}_\phi = -\frac{1}{2}(\nabla_c \phi \nabla^c \phi + 2V(\phi)) = -\frac{1}{2}(g^{dc} \nabla_d \phi \nabla_c \phi + 2V(\phi)). \quad (\text{D.0.2})$$

The energy-momentum tensor for a matter field is defined to be, cf. [77], p. 455,

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\mathcal{L}_{\text{matter}} \sqrt{-g})}{\delta g_{\mu\nu}}, \quad (\text{D.0.3})$$

for the Lagrangian (D.0.2), we find that

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \left((\nabla^c \phi \nabla_c \phi + 2V(\phi)) \frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}} + \sqrt{-g} \frac{\delta(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi)}{\delta g^{\mu\nu}} \right) = \quad (\text{D.0.4})$$

$$= \frac{1}{\sqrt{-g}} \left((\nabla^c \phi \nabla_c \phi + 2V(\phi)) \left(-\frac{1}{2} \sqrt{-g} g_{\mu\nu} \right) + \nabla_\mu \phi \nabla_\nu \phi \sqrt{-g} \right). \quad (\text{D.0.5})$$

Thus the energy-momentum tensor of a scalar field is

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla_c\phi\nabla^c\phi + 2V(\phi)), \quad (\text{D.0.6})$$

or equivalently,

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{dc}\nabla_d\phi\nabla_c\phi + 2V(\phi)). \quad (\text{D.0.7})$$

Comparing this with the energy-momentum tensor of a perfect fluid,

$$T_{\mu\nu} = (\mu + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (\text{D.0.8})$$

where u_μ is the unit timelike vector ($u^\mu u_\mu = -1$), it follows that

$$p = \mathcal{L}_\phi = -\frac{1}{2}(g^{dc}\nabla_d\phi\nabla_c\phi + 2V(\phi)), \quad (\text{D.0.9})$$

and

$$(\mu + p)u_\mu u_\nu = \nabla_\mu\phi\nabla_\nu\phi. \quad (\text{D.0.10})$$

Contracting the above equation, we find [99]

$$-(\mu + p) = \nabla^\nu\phi\nabla_\nu\phi = -2\mathcal{L}_\phi - 2V(\phi), \quad (\text{D.0.11})$$

and substituting for the pressure from Eq. (D.0.9) leads to

$$\mu = \mathcal{L}_\phi + 2V(\phi). \quad (\text{D.0.12})$$

In a homogeneous universe, a scalar field is a function of time only and hence Eqs. (D.0.9), (D.0.12) give the standard result,

$$p = \frac{\dot{\phi}^2}{2} - V(\phi), \quad (\text{D.0.13})$$

$$\mu = \frac{\dot{\phi}^2}{2} + V(\phi). \quad (\text{D.0.14})$$

The equation of motion of a scalar field can be found either by using the conservation law of energy, i.e., $\nabla^\nu T_{\mu\nu} = 0$, or by variation of the action S_ϕ with respect to ϕ . We use the second approach. We have that:

$$\frac{\delta\mathcal{L}_\phi}{\delta\phi} = \frac{\partial\mathcal{L}_\phi}{\partial\phi} - \left(\frac{\partial\mathcal{L}_\phi}{\partial(\phi_{;\mu})} \right)_{;\mu} = 2V'(\phi) - 2\nabla_c\nabla^c\phi, \quad (\text{D.0.15})$$

and so $\delta\mathcal{L}_\phi/\delta\phi = 0$ gives directly

$$\square\phi - V'(\phi) = 0, \quad (\text{D.0.16})$$

where $\square\phi = (-g)^{-1/2}\nabla_d((-g)^{1/2}g^{dc}\nabla_c\phi)$.

Appendix E

The method of asymptotic splittings

Consider a system of n first order ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{E.0.1}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))$. The *general solution* of the above system contains n arbitrary constants and hence describes the behaviour of the system for arbitrary initial data. A *particular solution* on the other hand contains less than n arbitrary constants and describes the behaviour of the system for a subset of initial data.

We say that a general or particular solution of the above dynamical system exhibits a finite-time singularity if there exists a $t_s \in \mathbb{R}$ and a $\mathbf{x}_0 \in \mathbb{R}^n$ such that

$$\lim_{t \rightarrow t_s} \|\mathbf{x}(t; \mathbf{x}_0)\| \rightarrow \infty, \tag{E.0.2}$$

where $\|\cdot\|$ is any L^p norm.

The purpose of *singularity analysis* (cf. [76], [211], pp. 129-155) is to build series expansions of solutions around the presumed position of a singularity at t_s in order to study the different asymptotic behaviours of the solutions of the system (E.0.1).

In particular, we look for series expansions of solutions that take the form of a Puiseux

series,

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\mathbf{a} + \sum_{i=1}^{\infty} \mathbf{c}_i \tau^{i/s} \right), \quad (\text{E.0.3})$$

where $\tau = t - t_s$, $\mathbf{p} \in \mathbb{Q}^n$, $s \in \mathbb{N}$.

The method of asymptotic splittings for the system (E.0.1) is realised in the following steps¹:

- First, we find all the possible *weight-homogeneous decompositions* of the vector field \mathbf{f} by splitting it into components $\mathbf{f}^{(j)}$:

$$\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)} + \dots + \mathbf{f}^{(k)}, \quad (\text{E.0.4})$$

with each of these components being *weight homogeneous*, that is to say

$$\mathbf{f}^{(j)}(\mathbf{a}\tau^{\mathbf{p}}) = \tau^{\mathbf{p} + \mathbf{1}(q^{(j)} - 1)} \mathbf{f}^{(j)}(\mathbf{a}) \quad j = 0, \dots, k, \quad (\text{E.0.5})$$

where $\mathbf{a} \in \mathbb{R}^n$ and $q^{(j)}$ are the positive non-dominant exponents that are defined by (E.0.7) below.

- We then substitute the forms $\mathbf{x} = \mathbf{a}\tau^{\mathbf{p}}$ in the system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ in order to find all possible *dominant balances*, i.e., sets of the form $\{\mathbf{a}, \mathbf{p}\}$. The *order* of each balance is defined by the number of the non-zero components of \mathbf{a} .

- For each of these balances we check the validity of the following *dominance condition*

$$\lim_{\tau \rightarrow 0} \frac{\sum_{j=1}^k \mathbf{f}^{(j)}(\mathbf{a}\tau^{\mathbf{p}})}{\tau^{\mathbf{p}-1}} = 0, \quad (\text{E.0.6})$$

and define the non-dominant exponents $q^{(j)}$, $j = 1, \dots, k$ by the requirement

$$\frac{\sum_{j=1}^k \mathbf{f}^{(j)}(\tau^{\mathbf{p}})}{\tau^{\mathbf{p}-1}} \sim \tau^{q^{(j)}}. \quad (\text{E.0.7})$$

The balances that cannot satisfy the condition (E.0.6) are then discarded.

¹Here we will refer briefly to the basic steps of the method of asymptotic splittings. A detailed analysis can be found in [76], [212].

- Next, we compute the Kovalevskaya matrix \mathcal{K} defined by

$$\mathcal{K} = D\mathbf{f}^{(0)}(\mathbf{a}) - \text{diag}\mathbf{p}, \quad (\text{E.0.8})$$

where $D\mathbf{f}^{(0)}(\mathbf{a})$ is the Jacobian matrix of $\mathbf{f}^{(0)}$ evaluated on \mathbf{a} .

- We calculate the spectrum of the \mathcal{K} -matrix, $\text{spec}(\mathcal{K})$, that is the set of its n eigenvalues, called also the \mathcal{K} -exponents. The arbitrary constants of any particular or general solution first appear in those terms in the series (E.0.3) whose coefficients \mathbf{c}_k have indices $k = \rho s$, where ρ is a non-negative \mathcal{K} -exponent and s is the least common multiple of the denominator of the set consisting of the non-dominant exponents $q^{(j)}$ and of the positive \mathcal{K} -exponents (cf. [76], [212]). The number of non-negative \mathcal{K} -exponents equals therefore the number of arbitrary constants that appear in the series expansions of (E.0.3). There is always the -1 exponent that corresponds to an arbitrary constant that is the position of the singularity at t_s . A dominant balance corresponds thus to a general solution if it possesses $n - 1$ non-negative \mathcal{K} -exponents (the n th arbitrary constant is the position of the singularity, t_s).

- Next, we substitute the Puiseux series:

$$x_i = \sum_{j=0}^{\infty} c_{ji} \tau^{p_i + j/s}, i = 1, \dots, n, \quad (\text{E.0.9})$$

in the system (E.0.1).

- We find the coefficients \mathbf{c}_j by solving the recursion relations

$$\mathcal{K}\mathbf{c}_j - \frac{j}{s}\mathbf{c}_j = \mathbf{P}_j(\mathbf{c}_1, \dots, \mathbf{c}_{j-1}) \quad (\text{E.0.10})$$

where \mathbf{P}_j are polynomials that are read off from the original system.

- We verify that for every $j = \rho s$, with ρ a positive \mathcal{K} -exponent, the following compatibility conditions hold:

$$\mathbf{v}^\top \cdot \mathbf{P}_j = 0, \quad (\text{E.0.11})$$

where v is the eigenvector associated with the positive \mathcal{K} -exponent ρ .

- We repeat the procedure for each possible decomposition.

We note here that if the compatibility condition is violated in some eigenvalue then the Puiseux series is not valid and instead we have to substitute a ψ -series, which is a series of the form

$$\mathbf{x} = \tau^{\mathbf{P}} \left(\mathbf{a} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{c}_{ij} \tau^{i/s} (\tau^{\rho} \log \tau)^{j/s} \right), \quad (\text{E.0.12})$$

where ρ is the first \mathcal{K} -exponent for which the compatibility condition is violated. The rest of the procedure in this case is the same as before.

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